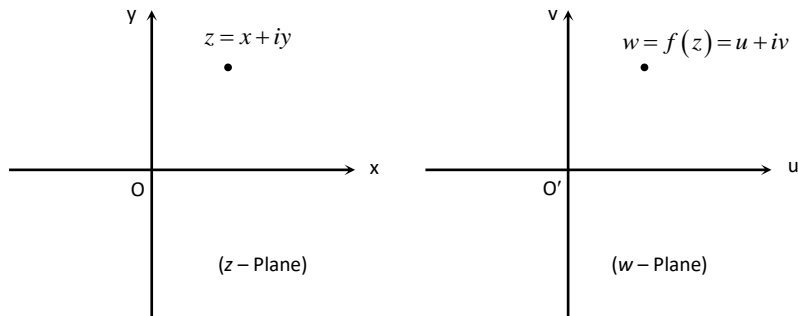


## Session\_3

### Function of a Complex Variable



Let  $z = x + iy = (x, y)$  be a complex variable.

If to each value of  $z$  in a region  $D$ , there we get one or more than one values of  $w = f(z) = u(x, y) + iv(x, y)$ , then  $w = f(z)$  is called a complex function/ complex valued function of  $z$ , where  $\operatorname{Re} f(z) = u(x, y)$  and  $\operatorname{Im} f(z) = v(x, y)$

**Limit:** A function  $f(z)$  is said to have the limit  $l$  as  $z \rightarrow z_0$ , written as

$$\lim_{z \rightarrow z_0} f(z) = l$$

iff for given  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $|f(z) - l| < \varepsilon$ , whenever  $|z - z_0| < \delta$

**Continuity:** The function  $f(z)$  is said to be continuous at  $z = z_0$  iff  $f(z_0)$  is defined and  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$ .  $f(z)$  is continuous in a domain iff it is continuous at each and every point of this domain.

**Derivative:** The derivative of a complex valued function  $f(z)$  at a point  $z_0$  is defined by

$$f'(z) \Big|_{z=z_0} = f'(z_0) = \operatorname{Lt}_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\text{or, } f'(z_0) = \operatorname{Lt}_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

### Analytic Function

If a single valued function  $f(z)$  is defined and differentiable (possesses a unique derivative) at every point of a region  $S$  except for a finite number of points then it is said to be an **analytic function** or a **regular function** or a **holomorphic function** or a **monogenic function** of  $z$  in  $S$ .

A point where the function ceases to be analytic is called a **singular point**.

If a function is analytic at each point in the entire finite plane (complex plane), then it is said to be an **entire function**.

### Cauchy-Riemann Equations

The Cauchy – Riemann equations are the most important equations and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function.

$$f(z) = f(x, y) = u(x, y) + iv(x, y)$$

Let  $f(z)$  be a single valued function defined in a region  $S$ . In order that  $f(z)$  is to be analytic in  $S$  or  $f'(z)$  exists it is necessary that  $u$  &  $v$  should satisfy the following pair of equations at every point of  $S$

The equations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

**Theorem:** The Complex-valued function  $w = f(z) = u + iv$  is analytic in a domain  $D$  iff C-R equations  $u_x = v_y$  &  $u_y = -v_x$  are satisfied, assuming all partial derivatives  $u_x, u_y, v_x,$  &  $v_y$  are continuous in  $D$ .