

Session_4

Proof of Theorem (CAUCHY-RIEMANN EQUATIONS)

Theorem: The Complex-valued function $w = f(z) = u + iv$ is analytic in a domain D iff C-R equations $u_x = v_y$ & $u_y = -v_x$ are satisfied, assuming all partial derivatives $u_x, u_y, v_x,$ & v_y are continuous in D .

Proof: Necessary part:

Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic in domain D .

$\Rightarrow f(z)$ is defined and differentiable at all points of D .

$$\Rightarrow f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \quad (2)$$

exists in D & also unique.

Now this equation (2) is independent of the manner, where $\Delta z \rightarrow 0$. So we can take the following two cases:

Case I: Let $\Delta x \rightarrow 0$ & $\Delta y = 0$. (i.e. along the real axis where $\Delta z \rightarrow 0$)

Now from (1) we get

$$\begin{aligned} f'(z) &= \lim_{\Delta x + i\Delta y \rightarrow 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y} \\ f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) + iv(x + \Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} \quad (\text{as } \Delta y = 0) \\ &= \lim_{\Delta x \rightarrow 0} \left(\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} \right) + i \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{ie.} \quad \boxed{f'(z) = u_x + iv_x} \quad (3) \end{aligned}$$

Case II: Let $\Delta x = 0$ & $\Delta y \rightarrow 0$ (i.e. along the imaginary axis)

$$\begin{aligned} f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) + iv(x, y + \Delta y) - u(x, y) - iv(x, y)}{i\Delta y} \\ &= \frac{1}{i} \left[\lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \right] \end{aligned}$$

$$= \frac{1}{i} \left(\frac{\partial u}{\partial y} \right) + \frac{\partial v}{\partial y} = \frac{-i^2}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{or} \quad \boxed{f'(z) = v_y - iu_y} \quad (4)$$

From (3) & (4) we get

$$u_x + iv_x = v_y - iu_y$$

Comparing both sides we get

$$\boxed{u_x = v_y} \quad \& \quad \boxed{u_y = -v_x}. \text{ These two are called C-R equations.}$$

Sufficient Part

Let C-R equations be satisfied i.e. $u_x = v_y$ & $u_y = -v_x$ (assuming all partial derivatives u_x, u_y, v_x & v_y are continuous in domain D)

$$\text{Now } f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)$$

$$= \left(u(x, y) + \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y \right) + i \left(v(x, y) + \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y \right)$$

(Applying Taylor's theorem for functions of two variables and neglecting the terms containing higher degrees of Δx & Δy)

$$= u(x, y) + iv(x, y) + \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) \Delta x + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) \Delta y$$

$$= f(z) + (u_x + iv_x) \Delta x + (-v_x + iu_x) \Delta y$$

$$\text{or } f(z + \Delta z) - f(z) = u_x (\Delta x + i\Delta y) + v_x (i\Delta x - \Delta y)$$

$$\text{or } f(z + \Delta z) - f(z) = u_x \Delta z + iv_x (\Delta x + i\Delta y)$$

$$\text{or } f(z + \Delta z) - f(z) = u_x \Delta z + iv_x \Delta z$$

$$\text{or } \frac{f(z + \Delta z) - f(z)}{\Delta z} = u_x + iv_x$$

Taking the limit of both sides; when $\Delta z \rightarrow 0$

$$\lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = u_x + iv_x$$

or $f'(z) = u_x + iv_x$ exists & unique

$\Rightarrow f(z)$ is analytic in D

Cauchy-Riemann Equations in polar form

Let $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$ (where $r > 0$)

& Let $w = f(z) = f(re^{i\theta}) = u + iv$

ie. $u(r, \theta) + iv(r, \theta) = f(re^{i\theta})$ (5)

Taking the Partial derivative of (5) w.r.t. r

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = f'(re^{i\theta})e^{i\theta}$$
 (6)

Similarly, taking the Partial derivative of (5) w.r.t. θ

$$\begin{aligned} \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} &= f'(re^{i\theta})ir e^{i\theta} \\ &= ir(f'(re^{i\theta})e^{i\theta}) \\ &= ir\left(\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r}\right) \quad [\text{using (6)}] \end{aligned}$$

or, $u_\theta + iv_\theta = -rv_r + iru_r$

$\Rightarrow u_\theta = -rv_r$ or $\boxed{v_r = -\frac{1}{r}u_\theta}$ (7) Again $v_\theta = ru_r$ or $\boxed{u_r = \frac{1}{r}v_\theta}$

(8) Equation (7) & (8) are called C-R equations in polar form.