

Session_5

Laplace's Equation, Harmonic and conjugate harmonic Functions

Theorem 1:

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D , then u & v satisfy Laplace's equation.

$$\text{i.e. } \nabla^2 u = u_{xx} + u_{yy} = 0$$

$$\& \quad \nabla^2 v = v_{xx} + v_{yy} = 0$$

respectively in D and both have continuous 2nd order partial derivative in D .

Proof:

Let given $f(z) = u(x, y) + iv(x, y)$ be analytic

$$\Rightarrow u_x = v_y \quad \& \quad u_y = -v_x \quad (\text{by C-R equations})$$

$$\text{So } u_{xx} = v_{yx} \quad \& \quad u_{yy} = -v_{xy}$$

Laplace's Equation:

Let $u(x, y)$ be a real valued function of two variables possessing 2nd order partial derivatives in a domain D .

If $\nabla^2 u = u_{xx} + u_{yy} = 0$, then this equation is known as Laplace's equation and $u(x, y)$ is said to satisfy the Laplace's equation.

$$\text{so, } \boxed{\nabla^2 u = u_{xx} + u_{yy} = 0} \Rightarrow \nabla^2 u \text{ satisfies Laplace's equation}$$

$$\text{Similarly } v_x = -u_y \Rightarrow v_{xx} = -u_{yx}$$

$$\& \quad v_y = u_x \Rightarrow v_{yy} = u_{xy}$$

$$\therefore \nabla^2 v = v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$$

$$\text{or } \boxed{\nabla^2 v = v_{xx} + v_{yy} = 0}$$

$\Rightarrow v$ satisfies Laplace's equation.

Harmonic function:

The function $u(x, y)$ is said to be harmonic if it satisfies Laplace's equation i.e. $\nabla^2 u = u_{xx} + u_{yy} = 0$.

Theorem 2: If $f(z) = u(x, y) + iv(x, y)$ is an analytic function then its real and imaginary parts i.e. u & v are harmonic.

Proof:

Since $f(z) = u + iv$ is analytic so it satisfies the C-R equations.

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (9)$$

$$\& \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (10)$$

Differentiating (9) partially w.r.t. x & (10) partially w.r.t y ; we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad (11)$$

And
$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad (12)$$

Assuming that partial derivatives of u & v of 1st and 2nd orders exist and are continuous functions of x & y ; it follows that;

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

Adding (3) & (4) we obtain

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad (13)$$

By differentiating (9) partially w.r.t. y and (10) partially w.r.t. x and subtracting, we obtain:

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0} \quad (14)$$

From equation (5) & (6) we concluded that the real and imaginary parts are u & v of an analytic function $f(z)$ satisfy the Laplace's equation

Hence u & v are known as Harmonic functions.