

Session_6

Harmonic function

The function $u(x, y)$ is said to be harmonic if it satisfies Laplace's equation i.e. $\nabla^2 u = u_{xx} + u_{yy} = 0$.

Theorem 2: If $f(z) = u(x, y) + iv(x, y)$ is an analytic function then its real and imaginary parts i.e. u & v are harmonic.

Proof:

Since $f(z) = u + iv$ is analytic so it satisfies the C-R equations.

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (9)$$

$$\& \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (10)$$

Differentiating (9) partially w.r.t. x & (10) partially w.r.t y ; we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad (11)$$

$$\text{And} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad (12)$$

Assuming that partial derivatives of u & v of 1st and 2nd orders exist and are continuous functions of x & y ; it follows that;

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

Adding (3) & (4) we obtain

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad (13)$$

By differentiating (9) partially w.r.t. y and (10) partially w.r.t. x and subtracting, we obtain:

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0} \quad (14)$$

From equation (5) & (6) we concluded that the real and imaginary parts are u & v of an analytic function $f(z)$ satisfy the Laplace's equation

Hence u & v are known as Harmonic functions.

Conjugate Harmonic function: If two harmonic functions u & v satisfy Cauchy-Riemann equations in a region S , then they are the real and imaginary parts of an analytic function $f(z)$ in S , i.e. $f(z) = u + iv$ is analytic in S .

So, v is called the 'conjugate harmonic function' of u in the region S . It is true that a harmonic conjugate, when it exists, is unique except for an additive constant.