

Session_8

Theorem: Cauchy's Second Integral Theorem/Cauchy's Integral Formula

If $f(z)$ be a single valued and analytic function throughout a simply connected domain D , then for any point a in D and for every simple closed path C in D that encloses the point a .

$$\oint_C \frac{f(z) dz}{z-a} = 2\pi i f(a) \quad (\text{Counter clockwise})$$

Proof:

The function $\left(\frac{f(z)}{z-a}\right)$ is analytic at all points in C , except at the point $z=a$. With the point a as centre and radius r draw a small circle C_1 lying inside the curve C .



$$\text{Thus } \oint_C \frac{f(z)}{z-a} dz = \oint_{C_1} \frac{f(z)}{z-a} dz$$

Now for any part z on C_1 , the equation of the circle C_1 is

$$|z-a| = r \Rightarrow z-a = re^{i\theta}, \text{ where } 0 \leq \theta \leq 2\pi$$

$$\Rightarrow z = a + re^{i\theta}$$

$$dz = rie^{i\theta} d\theta$$

$$\begin{aligned}
 \text{So } \oint_C \frac{f(z)}{z-a} dz &= \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\
 &= i \int_0^{2\pi} f(a+re^{i\theta}) d\theta \quad (2)
 \end{aligned}$$

In the limiting form as the circle C_1 shrinks to the point a i.e. $r \rightarrow 0$, the integral of (2) reduces to

$$= i \int_0^{2\pi} \left[\lim_{r \rightarrow 0} f(a+re^{i\theta}) \right] d\theta = i \int_0^{2\pi} f(a) d\theta = 2\pi i f(a)$$

$$\text{Thus } \oint_C \frac{f(z)}{z-z_0} dz = 2\pi i f(a) \text{ (counter clockwise)}$$