Session_9

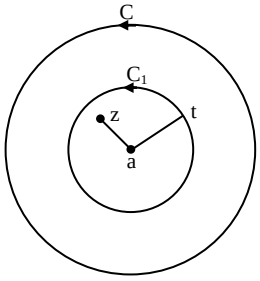
Taylor's Series of a Complex Function

If the function $f(z)$ is analytic inside a circle *C* with centre at *a*, then for all *z* inside *C*,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \ldots$$

Proof:

Let *z* be any point inside the circle *C*. A circle C_1 is drawn with centre at *a* and radius smaller than that of *C* such that *z* is an interior point of C_1



Let '*t*' be any point on C_1, then

$$|z-a| < |t-a| \quad \text{i.e.,} \quad \left|\frac{z-a}{t-a}\right| < 1$$

Now, $\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} = \frac{1}{t-a}\left[1-\frac{z-a}{t-a}\right]^{-1}$

Expanding R.H.S. by Binomial theorem as $\left|\frac{z-a}{t-a}\right| < 1;$ we obtain

$$\frac{1}{t-z} = \frac{1}{t-a}\left[1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^2+\cdots+\left(\frac{z-a}{t-a}\right)^n+\cdots\cdots\right] \qquad (2)$$

The series in R.H.S. of (2) converges uniformly since $\left|\frac{z-a}{t-a}\right| < 1$.

Now multiplying both sides of (1) by $\frac{f(t)}{2\pi i}$ and integrating term by term w.r.t *t* over the circle C_1, we obtain

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{t-z}dt = \frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{t-a}dt + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^2}dt$$

$$+\frac{(z-a)^2}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^3}dt + \cdots + \frac{(z-a)^n}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^{n+1}}dt + \cdots \qquad (3)$$

Since $f(t)$ is analytic on and inside C_1, so by Cauchy integral formula

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{t-z}dt = f(z)$$

and $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^{n+1}}dt$ or $\frac{1}{2\pi i}\oint_{C_1}\frac{f(t)}{(t-a)^{n+1}}dt = \frac{f^{(n)}(a)}{n!}$

$\therefore$ From (3) we obtain

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) \qquad (4)$$

which is the required Taylor's series for $f(z)$ about $z = a$.

Maclaurin Series

If $a = 0,$ then the Taylor,s series (4) is reduced to

$$f(z) = f(0) + zf'(0) + \frac{z^2}{2!} f''(0) + \cdots + \frac{z^n}{n!} f^n(0) + \cdots$$

which is known as Maclaurin's series expansion of *f(z)*.