



Centurion
UNIVERSITY
Shaping Lives...
Empowering Communities...

System Modeling & Control

Presented by

Prof. Amit Kumar Sahoo
CUTM, BBSR



Centurion
UNIVERSITY

Shaping Lives...
Empowering Communities...

Module IV

Stability in Time Domain



Centurion
UNIVERSITY
Shaping Lives...
Empowering Communities...

Stability of LCS

Total response = zero-input response + zero-state response

Lesson - 20

absolute stability

relative stability

Bounded-Input Bounded-output Stability

Def: (BIBO stable)

bounded $u(t) \Rightarrow$ bounded zero state response

$$y(t) = \int_0^\infty u(t-\tau)g(\tau)d\tau \quad \text{zero-state response}$$

$$|u(t)| \leq M \Rightarrow |y(t)| \leq \int_0^\infty |u(t-\tau)||g(\tau)|d\tau \leq M \int_0^\infty |g(\tau)|d\tau < \infty$$

FACT:

$$\text{BIBO stable} \Leftrightarrow \int_0^\infty |g(\tau)|d\tau < \infty$$

$$\Leftrightarrow \text{all poles of } G(s) \text{ are in the open LHP}$$

Zero-Input and Asymptotic Stability

Def: (zero-input stable or a.s.)

any finite initial conditions \Rightarrow bounded zero-input response
& $\lim_{t \rightarrow \infty} y(t) = 0$

FACT: Assume coprime transfer function.

a.s. stable \Leftrightarrow BIBO stable

Def: (marginally stable or marginally unstable)

any finite initial conditions \Rightarrow bounded zero-input response

FACT:

marginally stable \Leftrightarrow all poles of $G(s)$ are in the closed LHP
& $j\omega$ – axis poles are simple.

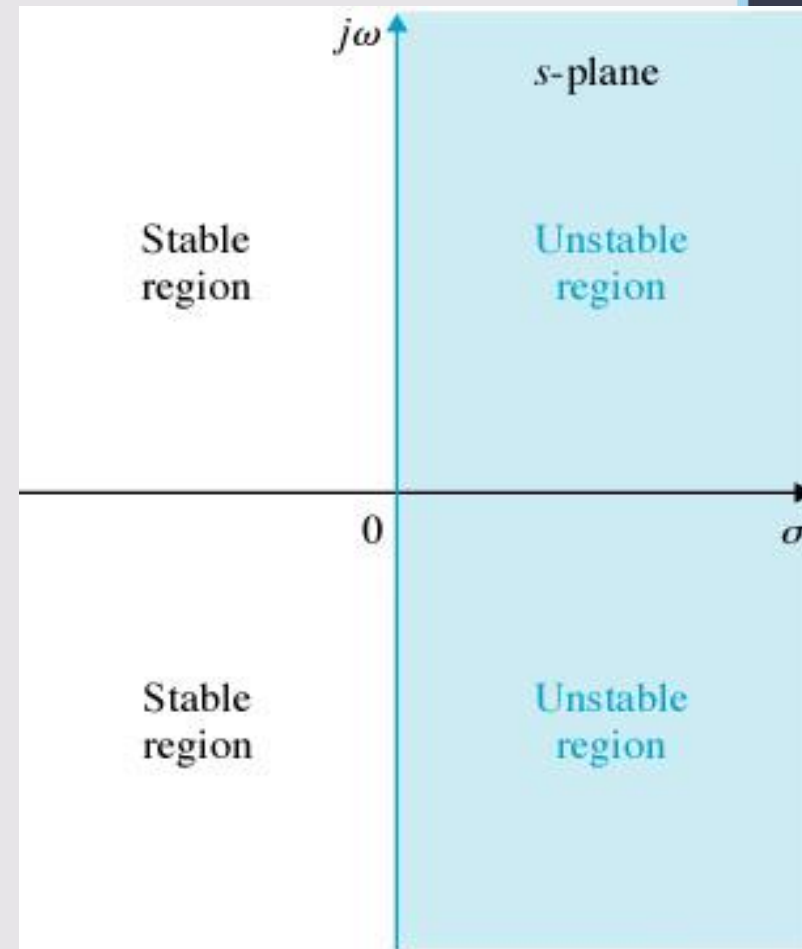


Centurion
UNIVERSITY
*Shaping Lives...
Empowering Communities...*

Zero-Input and Asymptotic Stability

Def: (unstable)

not marginaly stable \Rightarrow unstable





Zero-Input and Asymptotic Stability

Ex-1:-

$$M(s) = \frac{20}{(s+1)(s+2)(s+3)}$$

BIBO stable or a.s.

$$M(s) = \frac{20(s+1)}{(s-1)(s^2+2s+2)}$$

unstable

$$M(s) = \frac{20(s-1)}{(s+2)(s^2+4)}$$

marginally stable and NOT a.s.

$$M(s) = \frac{10}{(s^2+4)^2(s+10)}$$

unstable

$$M(s) = \frac{10}{s(s^3+30s^2+s+10)}$$

NOT a.s.



Centurion
UNIVERSITY
Shaping Lives...
Empowering Communities...

Methods of Determining Stability

Absolute stability : stable OR NOT?

1. Routh-Hurwitz criterion

Relative stability : how much stable?

2. Nyquist criterion

3. Bode diagram

4. Root-Locus

Routh-Hurwitz Criterion

- an algebraic method that provides the absolute stability

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$$

$$\frac{a_{n-1}}{a_n} = -\sum \text{all roots}$$

$$\frac{a_{n-2}}{a_n} = \sum \text{products of the roots taken two at a time}$$

...

$$\frac{a_{n-1}}{a_n} = (-1)^n \text{ products of all roots}$$

a.s. \Rightarrow all a_i 's have the same sign.

? **Routh-Hurwitz Criterion**

a.s. \Leftarrow all a_i 's have the same sign.

Ex:

$$(s+3)(s-1-j3)(s-1+j3) = (s+3)(s^2-2s+10) = s^3 + s^2 + 4s + 30$$

Routh's Tabulation

$$F(s) = s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

s^4	1	3	5	$A = \frac{2 \times 3 - 1 \times 4}{2} = 1$	$B = \frac{2 \times 5 - 1 \times 0}{2} = 5$
s^3	2	4		$C = \frac{1 \times 4 - 2 \times 5}{1} = -6$	$D = \frac{1 \times 0 - 2 \times 0}{1} = 0$
s^2	A	B			
s^1	C	D		$E = \frac{-6 \times 5 - 1 \times 0}{-6} = 5$	
s^0	E				

of sign change of 1st column = # of RHP roots

Routh-Hurwitz Criterion

Ex:

$$(s+3)(s-1-j3)(s-1+j3) = (s+3)(s^2-2s+10) = s^3 + s^2 + 4s + 30$$

Routh's Tabulation

s^3	1	4
s^2	1	30
s^1	$\frac{1 \times 4 - 1 \times 30}{1} = -26$	0
s^0	30	

2 sign change \Rightarrow 2 RHP roots

Routh-Hurwitz Criterion

Example- $2s^4 + s^3 + 3s^2 + 5s + 10 = 0$

Routh's Tabulation

$$s^4 \quad \quad \quad 2 \quad \quad \quad 3 \quad 10$$

$$s^3 \quad \quad \quad 1 \quad \quad \quad 5$$

$$s^2 \quad \frac{1 \times 3 - 2 \times 5}{1} = -7 \quad 10$$

$$s^1 \quad \frac{-7 \times 5 - 1 \times 10}{-7} = \frac{45}{7} \quad 0$$

$$s^0 \quad \quad \quad 10$$

2 sign change \Rightarrow 2 RHP roots

Routh-Hurwitz Criterion

Example- $s^4 + s^3 + 2s^2 + 2s + 3 = 0$

Routh's Tabulation

s^4	1	2	3
s^3	1	2	
s^2	$\frac{1 \times 2 - 1 \times 2}{1} = 0$	3	
s^1	$\frac{0 \times 2 - 1 \times 3}{0} = ?$	0	
s^0	??		

Routh-Hurwitz Criterion

Ex: $s^3 - 3s + 2 = 0$

Routh's Tabulation

s^3	1	-3
s^2	0	2
s^1	?	0
s^0	??	

Ex: $s^4 + 2s^2 + 3 = 0$

s^4	1	2	3
s^3	0	0	
s^2	?	?	
s^1	??	??	
s^0	???		

Routh-Hurwitz Criterion

Special cases when Routh's tabulation terminates prematurely

1. The first element in any one row of Routh's tab is zero, but the others are not.
2. The elements in one row of Routh's tab are all zero.

Remedy for the first case

$$s^3 - 3s + 2 = 0 \Rightarrow s^3 + \varepsilon s^2 - 3s + 2 = 0$$

s^3	1	-3
s^2	ε	2

$\varepsilon =$ positive very small number

s^1	$\frac{-3\varepsilon - 2}{\varepsilon} \approx \frac{-2}{\varepsilon}$	0
-------	--	---

2 sign change \Rightarrow 2 RHP roots

s^0	2
-------	---

Routh-Hurwitz Criterion

Example- $s^4 + s^3 + 2s^2 + 2s + 3 = 0$

Routh's Tabulation

s^4	1	2	3
s^3	1	2	
s^2	$0 \cong \varepsilon$	3	
s^1	$\frac{2\varepsilon - 3}{\varepsilon} \cong \frac{-3}{\varepsilon}$	0	
s^0	3		

2 sign change \Rightarrow 2 RHP roots

Routh-Hurwitz Criterion

2nd special cases occurs when roots are symmetric about the origin.

$$(s-2)(s+2) = s^2 - 4$$

$$(s-j2)(s+j2) = s^2 + 4$$

$$(s^2 + 4)^2 = s^4 + 8s^2 + 16$$

$$(s-1-j)(s-1+j)(s+1-j)(s+1+j) = (s^2 - 2s + 2)(s^2 + 2s + 2) = s^4 + 4$$

Remedy for the second case

auxiliary equation

$$A(s) = 0$$

$$s^2 \quad 1 \quad -4$$

$$s^1 \quad 0 \quad 0$$

$$s^0$$

$$\Rightarrow A(s) = s^2 - 4$$

Routh-Hurwitz Criterion

$$\Rightarrow \frac{dA(s)}{ds} = 2s \Rightarrow \begin{array}{ccc} s^2 & 1 & -4 \\ s^1 & 2 & 0 \\ s^0 & -4 & \end{array}$$

1 sign change \Rightarrow 1 RHP root

Example-

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

$$A(s) = 4s^2 + 4$$

$$\Rightarrow \frac{dA(s)}{ds} = 8s$$

$$\begin{array}{ccc} s^5 & 1 & 8 & 7 \\ s^4 & 4 & 8 & 4 \\ s^3 & \frac{32-8}{4} = 6 & \frac{28-4}{4} = 6 & 0 \\ s^2 & \frac{48-24}{6} = 4 & 4 & \\ s^1 & \frac{24-24}{4} = 0 & 0 & \end{array}$$

Routh-Hurwitz Criterion

s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	
s^1	8	0	
s^0	4		

NO sign change \Rightarrow NO RHP root

$$A(s) = 4s^2 + 4 = 0 \Rightarrow s = \pm j \quad \text{roots of original equation}$$

Routh-Hurwitz Criterion

Ex: $(s-1-j)(s-1+j)(s+1-j)(s+1+j) = (s^2 - 2s + 2)(s^2 + 2s + 2) = s^4 + 4$

$$\begin{array}{cccc} s^4 & 1 & 0 & 4 \\ s^3 & 0 & 0 & \end{array} \quad A(s) = s^4 + 4 \quad \Rightarrow \quad \frac{dA(s)}{ds} = 4s^3$$

$$\begin{array}{cccc} s^4 & 1 & 0 & 4 \end{array}$$

$$\begin{array}{cccc} s^3 & 4 & 0 & \end{array}$$

$$\begin{array}{cccc} s^2 & 0 \cong \varepsilon & 4 & \end{array}$$

$$\begin{array}{cccc} s^1 & \frac{-16}{\varepsilon} & 0 & \end{array}$$

$$\begin{array}{cccc} s^0 & 4 & & \end{array}$$

2 sign changes \Rightarrow 2 RHP roots

Routh-Hurwitz Criterion

Ex: $(s^2 + 4)^2 = s^4 + 8s^2 + 16$

$$\begin{array}{ccc} s^4 & 1 & 8 & 16 \\ s^3 & 0 & 0 & \end{array} \quad A(s) = s^4 + 8s^2 + 16 \Rightarrow \frac{dA(s)}{ds} = 4s^3 + 16s$$

$$\begin{array}{ccc} s^4 & 1 & 8 & 16 \\ s^3 & 4 & 16 & \end{array}$$

$$\begin{array}{ccc} s^2 & \frac{32-16}{4} = 4 & 16 \\ s^1 & \frac{64-64}{4} = 0 & 0 \end{array}$$

$$\begin{array}{ccc} s^1 & \frac{64-64}{4} = 0 & 0 \\ s^0 & & \end{array}$$

$$B(s) = 4s^2 + 16 \Rightarrow \frac{dB(s)}{ds} = 8s$$

Routh-Hurwitz Criterion

$$s^4 \quad 1 \quad 8 \quad 16$$

$$s^3 \quad 4 \quad 16$$

$$s^2 \quad 4 \quad 16$$

$$s^1 \quad 8 \quad 0$$

$$s^0 \quad 16$$

0 sign change \Rightarrow 0 RHP root

$$B(s) = 4s^2 + 16 = 0 \Rightarrow s = \pm j2 \quad \text{roots of original equation}$$

Routh-Hurwitz Criterion

Example- $s^3 + 3408.3s^2 + 1204000s + 1.5 \times 10^7 K = 0$

s^3	1	1204000
s^2	3408.3	$1.5 \times 10^7 K$
s^1	$\frac{410.36 \times 10^7 - 1.5 \times 10^7 K}{3408.3}$	0
s^0	$1.5 \times 10^7 K$	

$$\text{a.s.} \Rightarrow \frac{410.36 \times 10^7 - 1.5 \times 10^7 K}{3408.3} > 0 \quad \& \quad 1.5 \times 10^7 K > 0$$

$$0 < K < 273.57$$

Routh-Hurwitz Criterion

$$K = 273.57 \Rightarrow A(s) = 3408.3s^2 + 4.1036 \times 10^9 = 0$$

$$s = \pm j1097 \quad \text{roots when } K = 273.57$$

Routh-Hurwitz Criterion

Example- $s^3 + 3Ks^2 + (K + 2)s + 4 = 0$

s^3	1	$K + 2$	
s^2	$3K$	4	
s^1	$\frac{3K(K + 2) - 4}{3K}$	0	
s^0	4		

a.s. $\Rightarrow 3K > 0 \text{ \& } \frac{3K(K + 2) - 4}{3K} > 0$

$$3K^2 + 6K - 4 > 0 \Rightarrow K < -2.528 \text{ or } K > 0.528$$

a.s. $\Rightarrow K > 0.528$