

# Unit 15

## Shearing and Torsion (and Bending) of Shell Beams

### Readings:

|         |   |
|---------|---|
| Rivello | Ch. 9, section 8.7 (again), section 7.6 |
| T & G   | 126, 127                                |

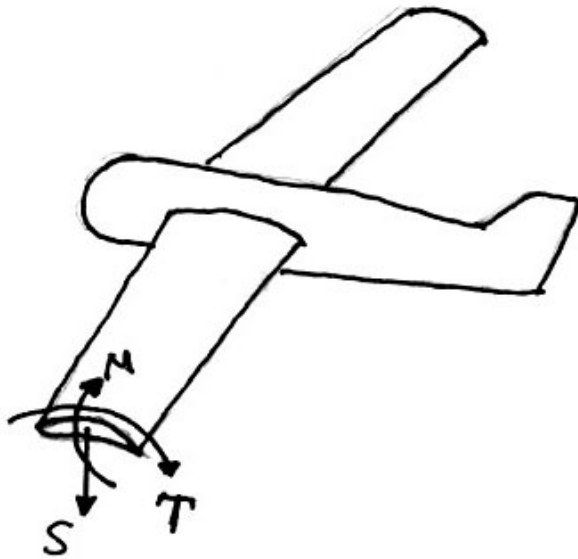
Paul A. Lagace, Ph.D.  
Professor of Aeronautics & Astronautics  
and Engineering Systems

Thus far, we have concentrated on the bending of shell beams. However, in the general case a beam is subjected to:

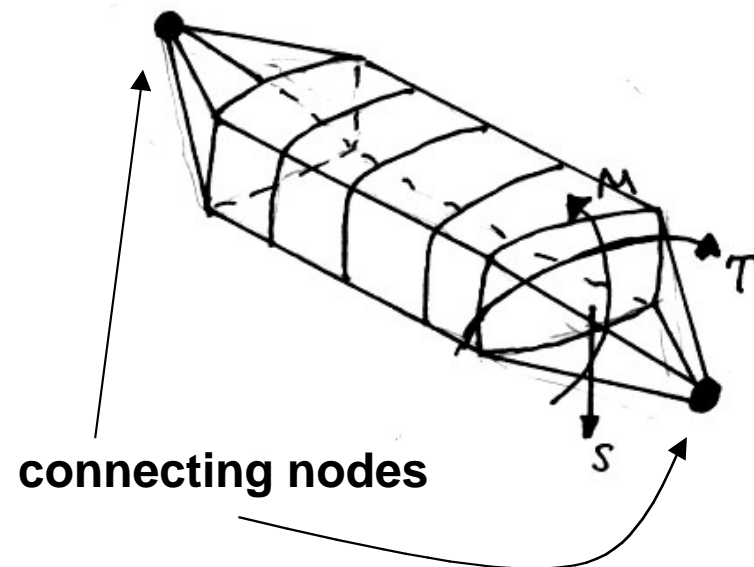
- axial load,  $F$
- bending moments,  $M$
- shear forces,  $S$
- torque (torsional moments),  $T$

**Figure 15.1** Examples of general aerospace shell beam structures

Airc



ell



Idealize the cross-section of the shell beam into two parts:

- Parts that carry extensional stress,  $\sigma_{xx}$  (and thus the bending and axial loads)
- Parts that carry shear stress  $\sigma_{xs}$  (and thus the shear loads and torques)

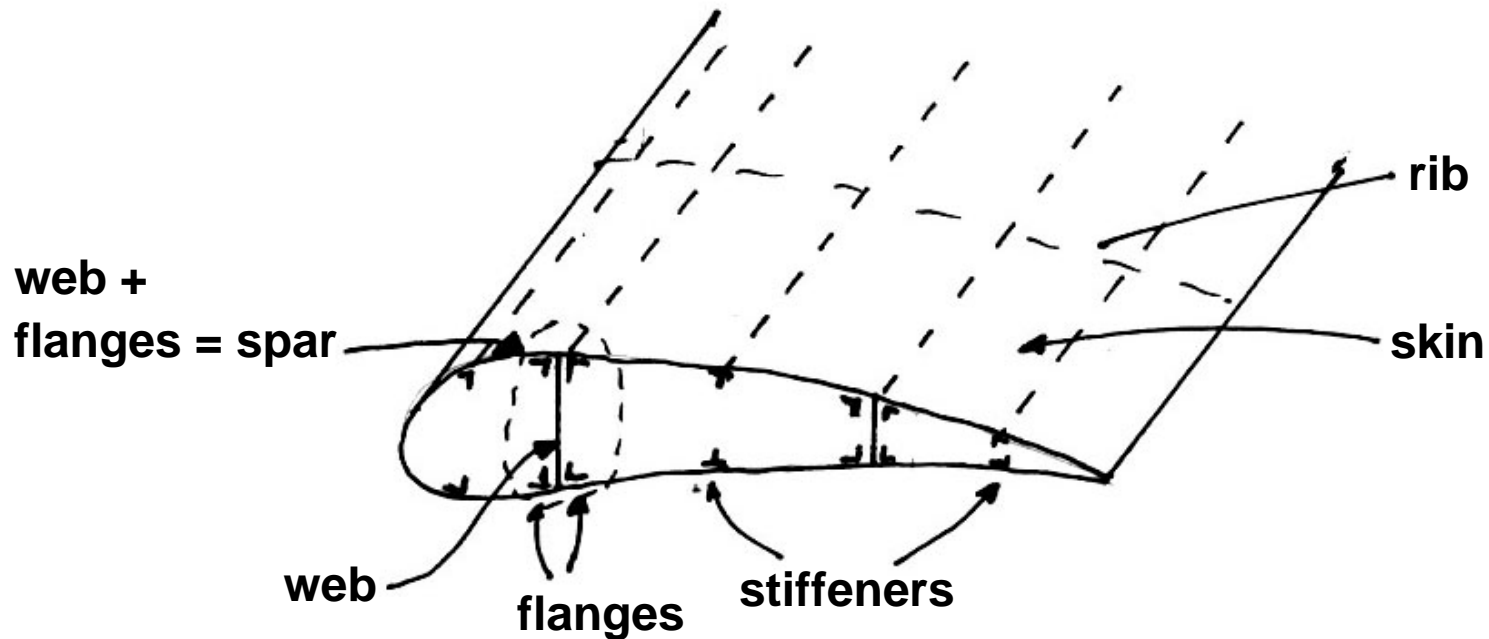
Two examples again...

- high aspect ratio wing with *semi-monocoque* construction

Notes:

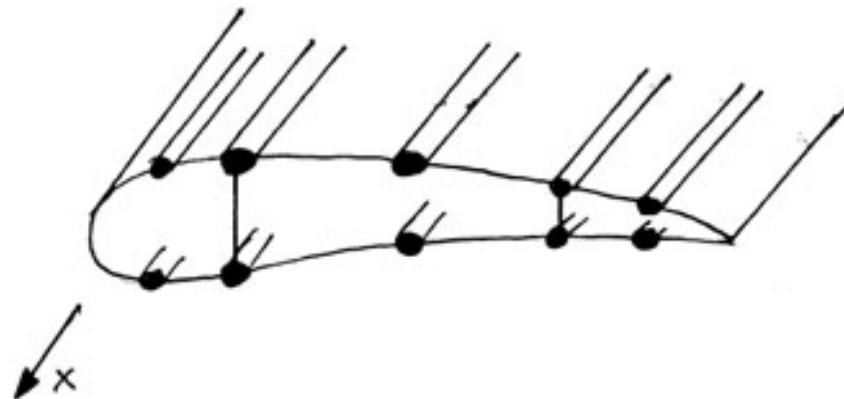
- *monocoque* construction
  - – all in one piece without internal framing
  - – from French “coque” meaning “eggshell”
  - – “mono” = one piece
- *semi-monocoque*
  - stressed skin construction with internal framework
  - still have “eggshell” to carry shear stresses,  $\sigma_{xs}$
  - internal framework to carry axial stress,  $\sigma_{xx}$

**Figure 15.2 Representation of wing semi-monocoque construction**



Idealize this section as:

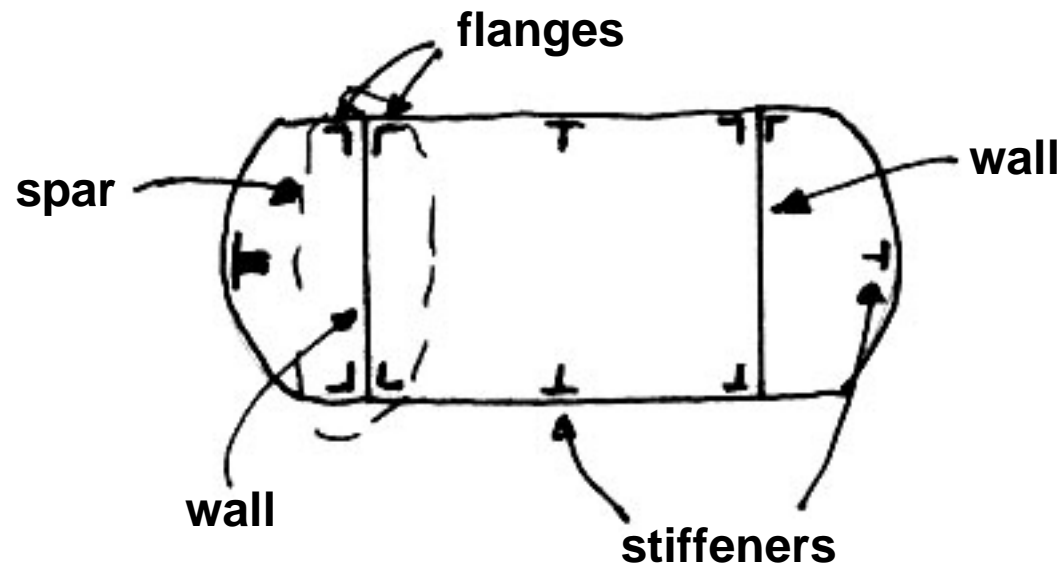
**Figure 15.3 Idealization of wing semi-monocoque construction**



- Skins and webs are assumed to carry only shear stress  $\sigma_{xs}$
- Flanges and stringers are assumed to carry only axial stress  $\sigma_{xx}$

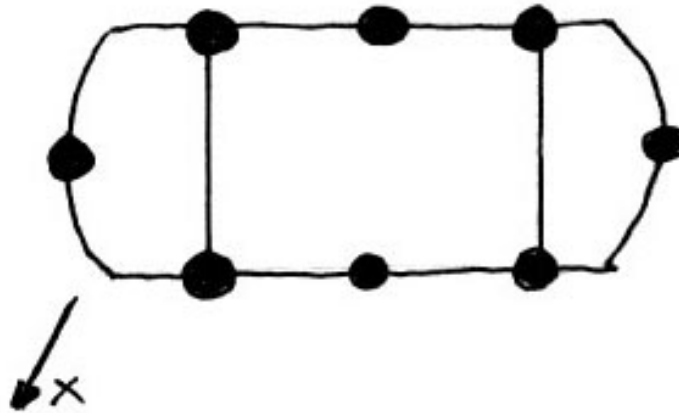
- Space habitat

**Figure 15.4** Representation of space habitat semi-monocoque construction



Idealize as for wing:

**Figure 15.5 Idealization of space habitat semi-monocoque construction**



- Outer skin and walls are assumed to carry only shear stress  $\sigma_{xs}$
- Flanges and stiffeners are assumed to carry only axial stress  $\sigma_{xx}$

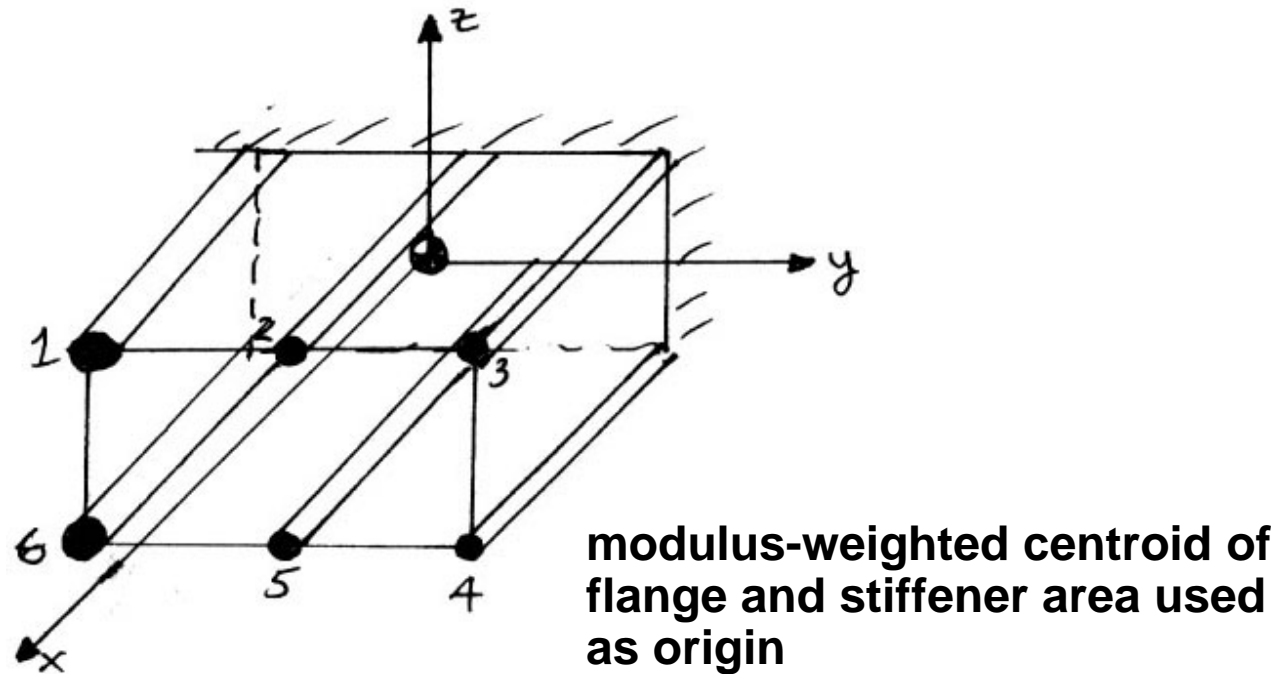
Analyze these cross-sections as a beam under combined bending, shear, and torsion. Utilize St. Venant assumptions:

1. There are enough closely spaced rigid ribs to preserve the shape of the cross-section (or enough stiffness in the internal bracing to do such)
2. The cross-sections are free to warp out-of-plane

Start to develop the basic equations by looking at the most basic case:

# Single Cell "Box Beam"

**Figure 15.6** Representation of geometry of single cell box beam



Breakdown the problem...

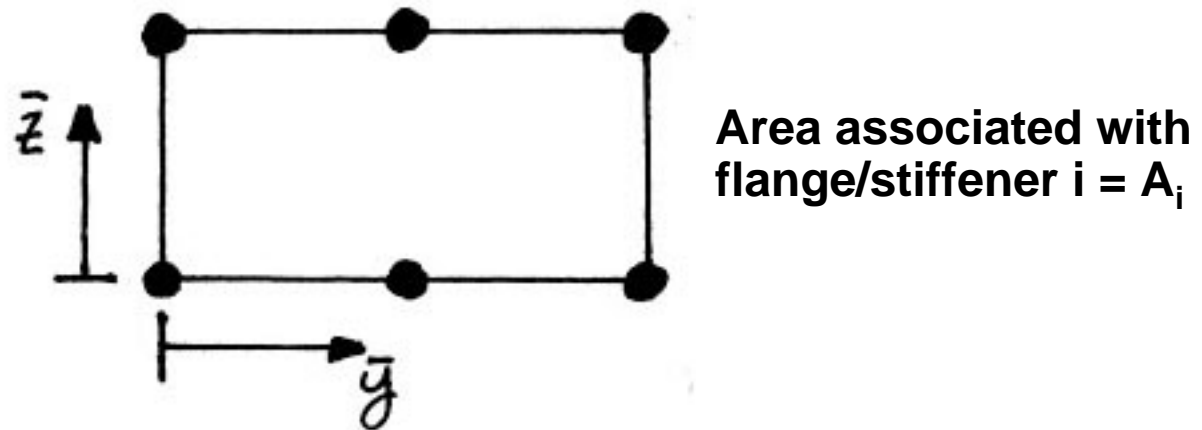
- (a) **Axial Bending Stresses**: Each flange/stiffener has some area associated with it and it carries axial stress only (assume  $\sigma_{xx}$  is constant within each flange/stiffener area)

The axial stress is due only to bending (and axial force if that exists -- *leave at zero for now*) and is therefore independent of the twisting since the wing is free to warp (except near root -- St. Venant assumptions)

\* Find  $M$ ,  $S$ ,  $T$  from statics at any cross-section  $x$  of the beam

Consider the cross-section:

**Figure 15.7 Representation of cross-section of box beam**



Find the modulus-weighted centroid (Note: flange/stiffeners may be made from different materials)



- Choose some axis system  $\bar{y}, \bar{z}$  (convenience says one might use a “corner” of the beam)
- Find the modulus-weighted centroid location:

$$\bar{y}^* = \frac{\sum A_i^* \bar{y}_i}{\sum A_i^*}$$

$$\bar{z}^* = \frac{\sum A_i^* \bar{z}_i}{\sum A_i^*}$$

(  $\sum_{i=1}^n$  = sum over number of flanges/stiffeners)  
number = n

(Note: If flanges/stiffeners are made of the same material, remove the asterisks)

- Find the moments of inertia with reference to the coordinate system with origin at the modulus-weighted centroid

$$I_y^* = \sum A_i^* z_i^{*2}$$

$$I_z^* = \sum A_i^* y_i^{*2}$$

$$I_{yz}^* = \sum A_i^* y_i^* z_i^*$$

- Find the stresses in each flange by using the equation previously developed:

$$\sigma_{xx} = \frac{E}{E_1} \left\{ \underbrace{\frac{F^{TOT}}{A^*}} - E_1 f_2 y - E_1 f_3 z - E_1 \alpha \Delta T \right\}$$

0 for no axial force

(Will do an example of this in recitation)

- (b) **Shear stresses**: assume the skins and webs are thin such that the shear stress is constant through their thickness.

Use the concept of “shear flow” previously developed:

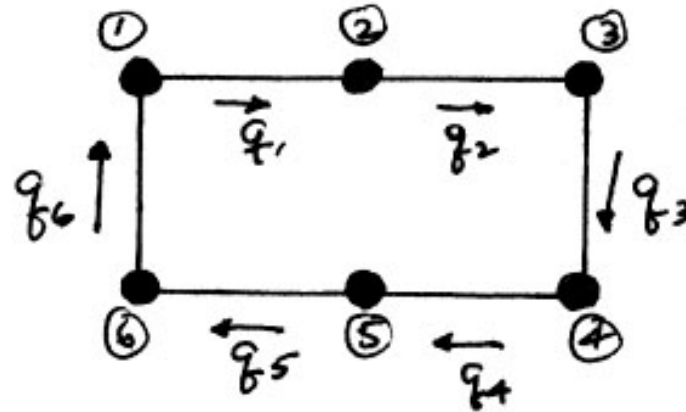
$$q = \sigma_{xs} t \quad [\text{Force/length}]$$

shear flow  $\nearrow$   $q$   $\leftarrow$  thickness  $\nearrow$   $t$

$\uparrow$   $\sigma_{xs}$  shear stress (called this the shear resultant in the case of torsion)

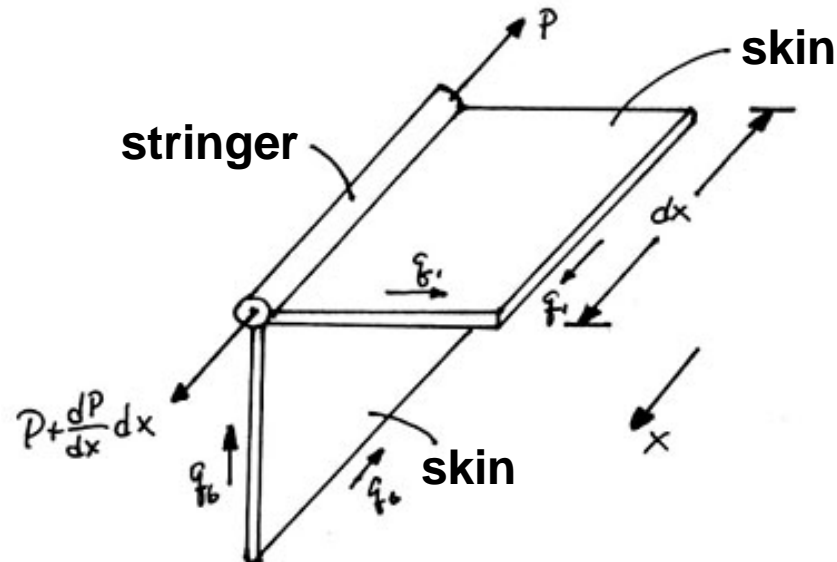
Look at the example cross-section and label the “joints” and “skins”

**Figure 15.8** Representation of joints, skins, and shear flows in cross-section of box beam



Look at the equilibrium of joint ①:

**Figure 15.9** Representation of skins and stringer and associated loads and shear flows at joint ①



Notes:

- The stringer only carries axial load
- The skin carries only shear flow
- The shear flow at the “end” of the skin (where it is “cut”) must be the same as at the edge (the cross-section cut). This is due to equilibrium ( $\sigma_{xy} = \sigma_{yx}$ )

- Apply equilibrium:  $\sum F_x = 0$

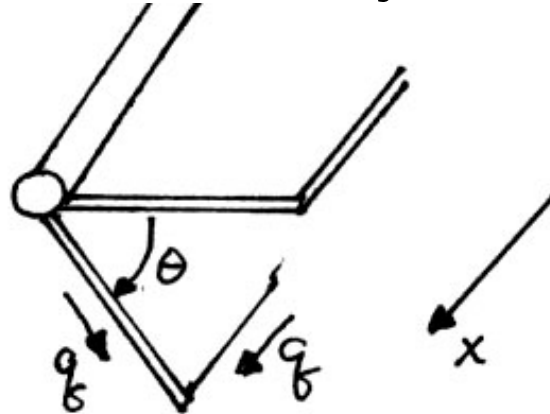
$$\Rightarrow -P + \Delta P + \Delta \frac{dP}{dx} dx + \Delta q_1 dx - q_6 dx = 0$$

$$\Rightarrow q_1 - q_6 = -\frac{dP}{dx}$$

More generally note that:

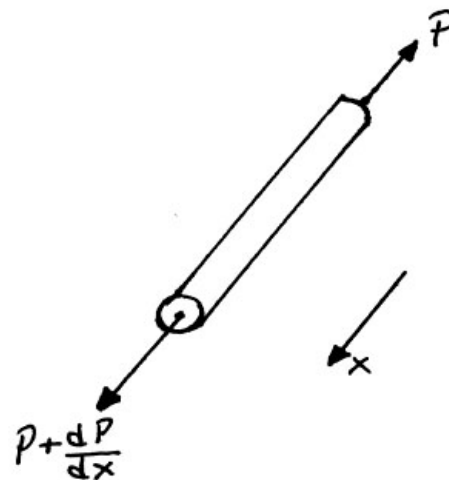
1. Angle that skin comes into joint doesn't matter since  $q$  along edge is always in  $x$ -direction

**Figure 15.10 Representation of skins at joint coming in at any angle**



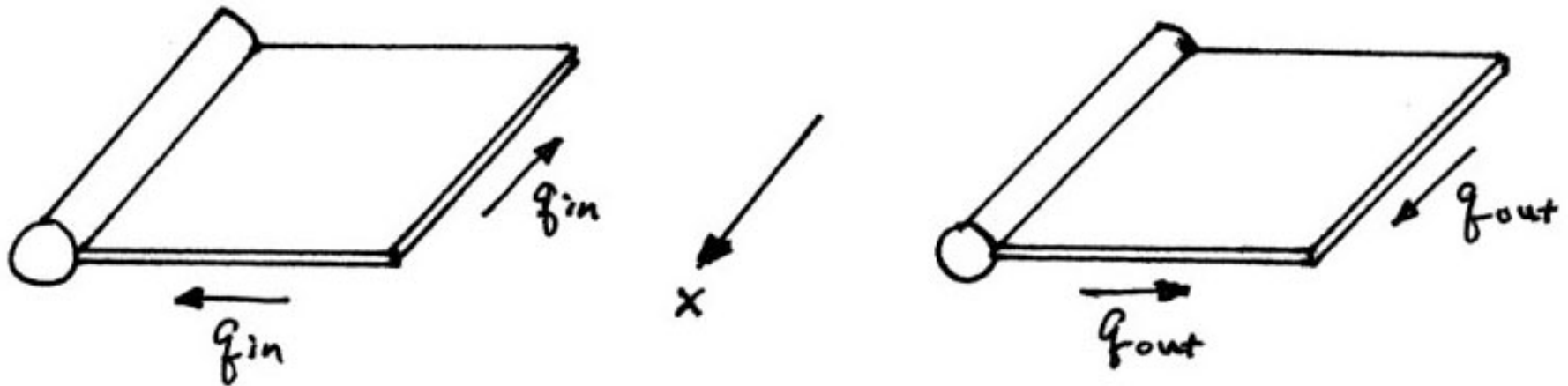
2. Stringer alone gives  $\frac{dP}{dx}$  as its contribution

**Figure 15.11 Representation of stringer isolated at joint**



3. If shear flows “into” joint, its contribution is in the *negative* x-direction; if shear flows “out of” joint, its contribution is in the *positive* x-direction

**Figure 15.12 Representation of shear flowing (left) *into* and (right) *out of* joint**



Adding all this up:

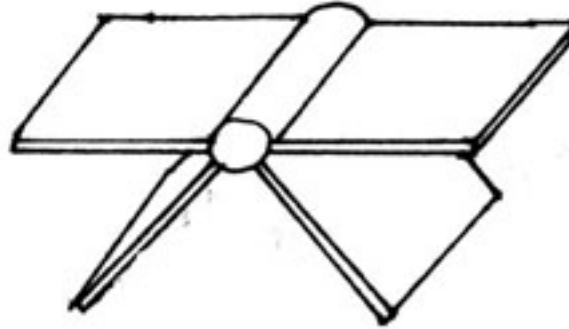
$$\frac{dP}{dx} - q_{in} + \square q_{out} = 0$$

$$\Rightarrow q_{out} - q_{in} = \square - \frac{dP}{dx}$$

Use this in general

For a more complicated joint, use superposition

**Figure 15.13 Representation of joint with multiple skins**



--> Need an expression for  $P$  -- start with:

$$P = \int A \sigma_{xx}$$

differentiating:

$$\Rightarrow \frac{dP}{dx} = \int A \frac{d\sigma_{xx}}{dx} + \int \underbrace{\sigma_{xx}}_{=0} \frac{dA}{dx}$$

= 0 since are considering stringers  
with a uniform cross-section

Most general case:

$$q_{out} - q_{in} = \int -A \frac{d\sigma_{xx}}{dx}$$

**Joint Equilibrium**

Consider a simpler case:

- $I_{yz} = 0$  (symmetric section)
- $M_z = 0$

know that:

$$\sigma_{xx} = -\frac{M_y z}{I_y}$$

$$\Rightarrow \frac{dP}{dx} = -A \frac{d}{dx} \left( -\frac{M_y z}{I_y} \right)$$

$$\frac{dP}{dx} = -\frac{A z}{I_y} \frac{dM_y}{dx}$$

Recall that:

$$\frac{dM_y}{dx} = S_z \quad (\text{shear resultant})$$

$$A z = Q_y \quad (\text{moment of area about } y)$$

So for this case, the joint equilibrium equation becomes:



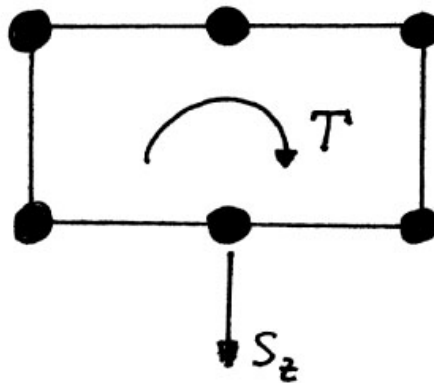
$$q_{out} - q_{in} = \frac{Q_y S_z}{I_y}$$

- Symmetric section
- $M_z = 0$

Now have an equation for the equilibrium of shear stresses at the joints. Shear stresses arise due to two reasons:

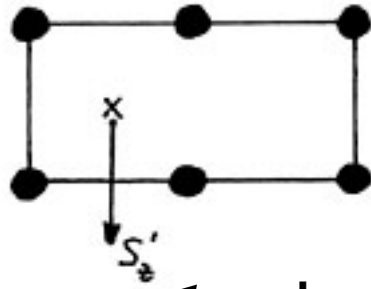
- Shear resultant
- Twisting

In general have at any cross-section:



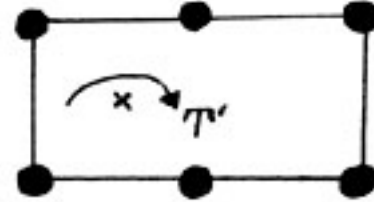
It is convenient to break up the problem into two separate problems:

(1) "Pure Shear"



+

(2) "Pure Twist"

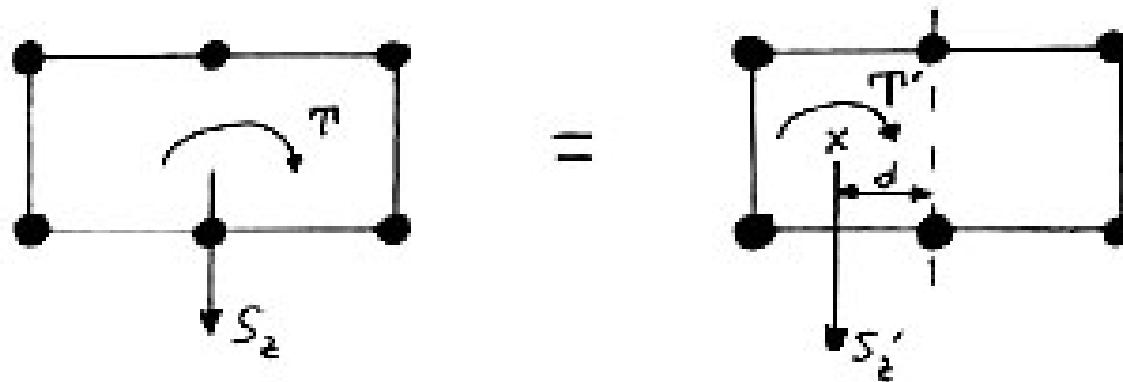


shear resultant acts  
at shear center so  
there is no twisting

◇ --> Solve each problem separately, then add the results (use superposition)

Condition: The two force systems ( $S_z$ ,  $T$  and  $S'_z$ ,  $T'$ ) must be **equipollent**

Figure 15.14 Demonstration of equipollence of force systems



**Define:**  $d$  = distance from where shear resultant acts to shear center

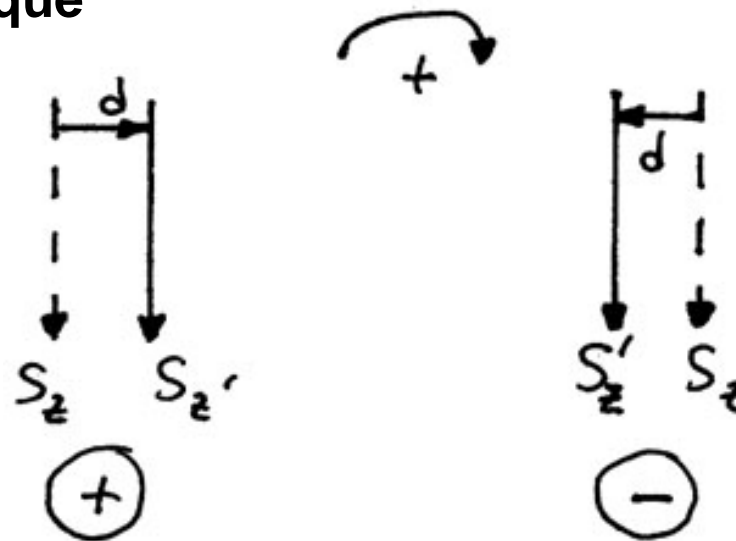
$$\sum F_z = \text{same} \Rightarrow S_z = S_z'$$

$$\sum T = \text{same} \Rightarrow T' - S_z d = T$$



careful: sign could be + or - depending upon direction  $S_z$  is moved!

**Figure 15.15** Representation of positive and negative contribution of  $S_z$  to torque to torque



Hint: Add up torques about line of  $S_z$  action in each case. They must be the same!  
 $(\Rightarrow d$  has magnitude and sign)

## Solution procedure

- Given:
- section properties
  - loading  $[T(x), S_z(x)]$

- Find:
- shear stresses (flows) (n joints)
  - shear center

$\Rightarrow (n + 1)$  variables

### 1. Consider “Pure Shear” case

- a) Apply joint equilibrium equation at each joint

Note: n joints will yield n-1 **independent** equations. (one is dependent since the section is closed)

- b) Use Torque Boundary Condition

$$\sum T_{\text{internal}} = T_{\text{applied}}$$

This is torque equivalence, **not** equilibrium

Do this about the line of action of  $S_z$

Then:

$$T_{\text{applied}} = S_z d$$

$$\sum T_{\text{internal}} = \sum q_i (\text{moment arm})_i (\text{skin length})_i$$

- c) Specified no twist (Pure Shear Case), so apply the **No Twist Condition**

Recall from Torsion Theory:

$$\oint \tau ds = 2AG \frac{d\alpha}{dx}$$

$$\text{Here: } \frac{d\alpha}{dx} = 0$$

$$\text{and: } \tau = \frac{q}{t}$$

$$\Rightarrow \boxed{\oint \frac{q}{t} ds = 0}$$

This gives:

- a)  $n - 1$  equations
  - b) 1 equation
  - c) 1 equation
- $n + 1$  equations for  $n + 1$  variables

Solving these gives:

- $q$ 's due to "Pure Shear" case
- $d$

--> when complete, check via:

$$\sum \text{Internal Shear loads} = \text{Applied Shear}$$

(Horizontal & vertical)

## 2. Consider "Pure Torsion" case

- a) Apply joint equilibrium equation at each joint
- Note: again,  $n$  joints give  $n-1$  equations

Since no shear:

$$q_{out} - q_{in} = 0$$

## b) Use Torque Boundary Condition

$$\sum T_{\text{internal}} = T_{\text{applied}}$$

$$T_{\text{applied}} = T + \text{or} - S_z d$$

↑  
found in part 1

This gives:

- a)  $n - 1$  equations
- b) 1 equation

- Need:  $n$   $q$ 's

⇒ Solve these for  $q$ 's due to "Pure Torsion" case

### 3. Sum results for "Pure Shear" and "Pure Torsion" cases

(could use  $q_i^S$ ,  $q_i^T$ :  $q_i$  due to pure shear =  $q_i^S$   
 $q_i$  due to pure torsion =  $q_i^T$ )

Now have: shear flows  
(get shear stresses via:  $\tau_{xs} = \frac{q}{t}$ )  
location of shear center  
(will be important to determine deflection)  
(will go over sample problem of handout #6 in recitation)

for: Unsymmetric Shell Beams

- Cannot make simplifying assumptions (use equations coupling bending in y and z)
- See handout #4B

Now that the stresses (due to bending, shear and torsion) are determined, proceed to find the...

## Deflections of Shell Beams

Have divided the problem and used superposition for the stresses. Again use the:

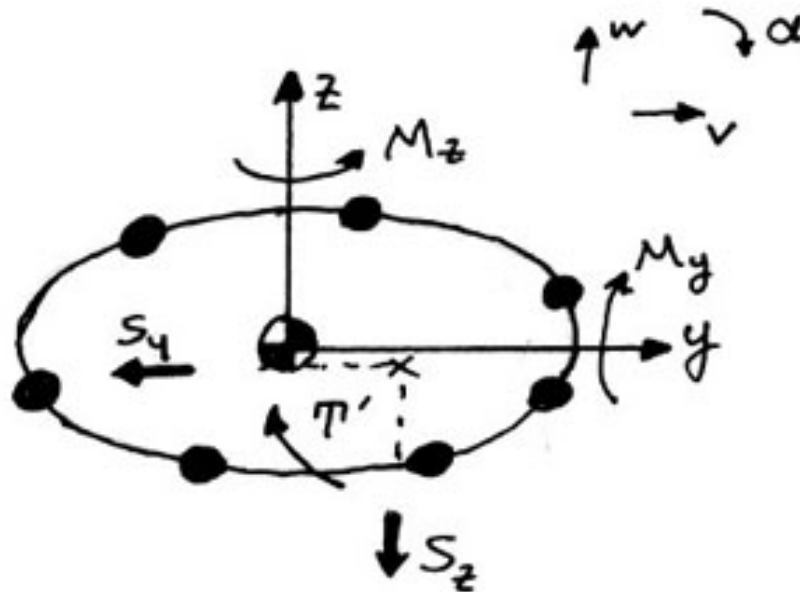
St. Venant Assumption: Bending and Torsion are independent



Thus, can add the effects of bending, shearing, and torsion deflections to get the total deflections.

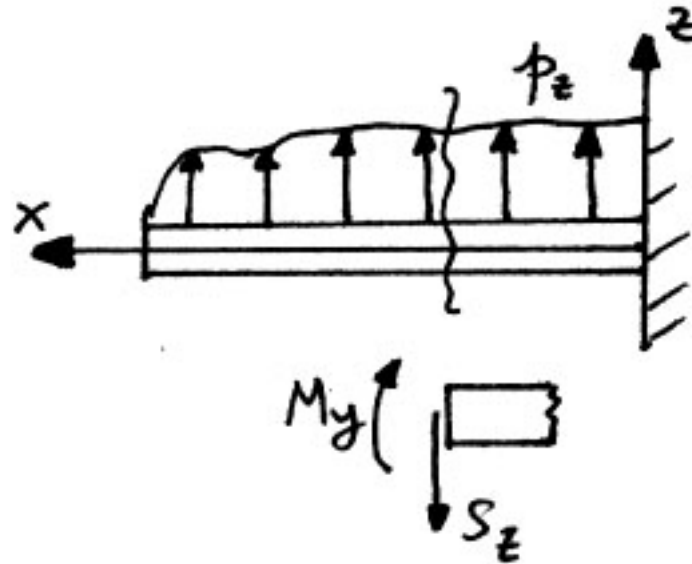
### General Approach:

**Figure 15.16 Representation of loading resultants and deflections for general cross-section**



- Place  $S_y$  and  $S_z$  at the shear center
- $T'$  acts about the shear center
- $v, w$  = deflection of shear center
- $\alpha$  = twist about shear center

### Figure 15.17 Representation of loading and resultants in x-z plane



- From the loadings  $p_y$ ,  $p_z$  obtain the resultants  $S_y$ ,  $S_z$ ,  $M_y$ ,  $M_z$ ,  $T$  as functions of  $x$
- Use the deflection equations previously developed:

for bending: (subscript "B" = bending)

$$\frac{d^2 v_B}{dx^2} = \frac{I_y M_z}{E (I_y I_z - I_{yz}^2)} - \frac{I_{yz} M_y}{E (I_y I_z - I_{yz}^2)}$$

$$\frac{d^2 w_B}{dx^2} = \frac{I_z M_y}{E (I_y I_z - I_{yz}^2)} - \frac{I_{yz} M_z}{E (I_y I_z - I_{yz}^2)}$$

for shearing: (subscript "s" = shearing)

$$\frac{dv_s}{dx} = -\frac{S_y}{GA_{yy}} - \frac{S_z}{GA_{yz}}$$

$$\frac{dw_s}{dx} = -\frac{S_z}{GA_{zz}} - \frac{S_y}{GA_{yz}}$$

for torsion/twisting

$$\frac{d\alpha}{dz} = \frac{T}{GJ}$$

then get:

$$V = V_B + V_s$$

$$W = W_B + W_s$$

$\alpha$  = rotation about center

In order to do this, need to know the...

## SECTION PROPERTIES

a) Bending Stiffness,  $EI$  (as before)

$$I_z = \sum A^* y^2$$

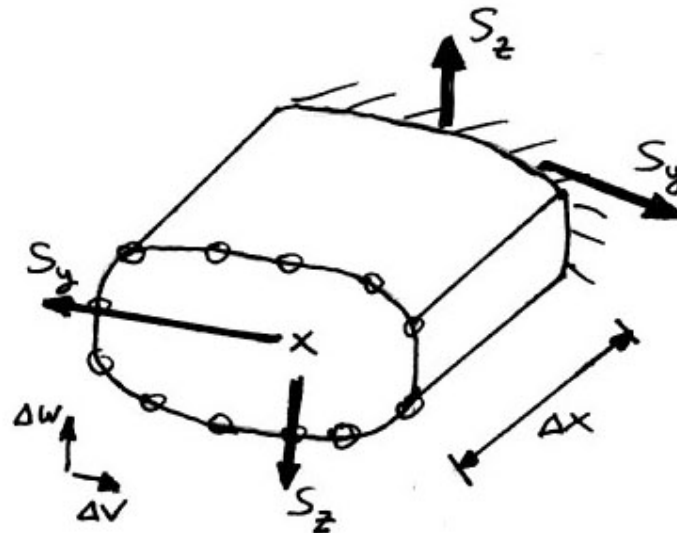
$$I_y = \sum A^* z^2$$

$$I_{yz} = \sum A^* yz$$

b) Shear Stiffness,  $GA$  (have not done this before)

Consider the deflections  $\Delta v$  and  $\Delta w$  for a segment  $\Delta x$  with only shear forces  $S_y$  and  $S_z$  acting at the shear center

**Figure 15.18 Representation of shell beam segment with only shear forces acting at the shear center**



Can express the shear flows in each member as contributions due to  $S_y$  and  $S_z$ :

$$q(s) = \bar{q}_y(s) S_y + \bar{q}_z(s) S_z$$

where:

$\bar{q}_y(s)$  = shear flow due to  $S_y$  of unit magnitude

$\bar{q}_z(s)$  = shear flow due to  $S_z$  of unit magnitude

--> To determine  $\Delta v$  and  $\Delta w$ , it is best to use an Energy Method

It can be shown that:

$$A_{yy} = \frac{1}{\int \frac{(\bar{q}_y)^2}{t} ds}$$

$$A_{zz} = \frac{1}{\int \frac{(\bar{q}_z)^2}{t} ds}$$

$$A_{yz} = \frac{1}{\int \frac{\bar{q}_y \bar{q}_z}{t} ds}$$

c) Torsional Stiffness,  $GJ$ 

Previously saw that:

$$\frac{d\alpha}{dx} = \frac{1}{2AG} \oint \frac{q}{t} ds \quad (\text{for closed section})$$

Let: (as just did)

$$q = \bar{\bar{q}}(s) T$$

where:

$\bar{\bar{q}}(s)$  = shear flow due to  $T$  of unit magnitude

Then, using this in the above:

$$\frac{d\alpha}{dx} = \frac{T}{G2A} \oint \frac{\bar{\bar{q}}}{t} ds$$

Recalling:

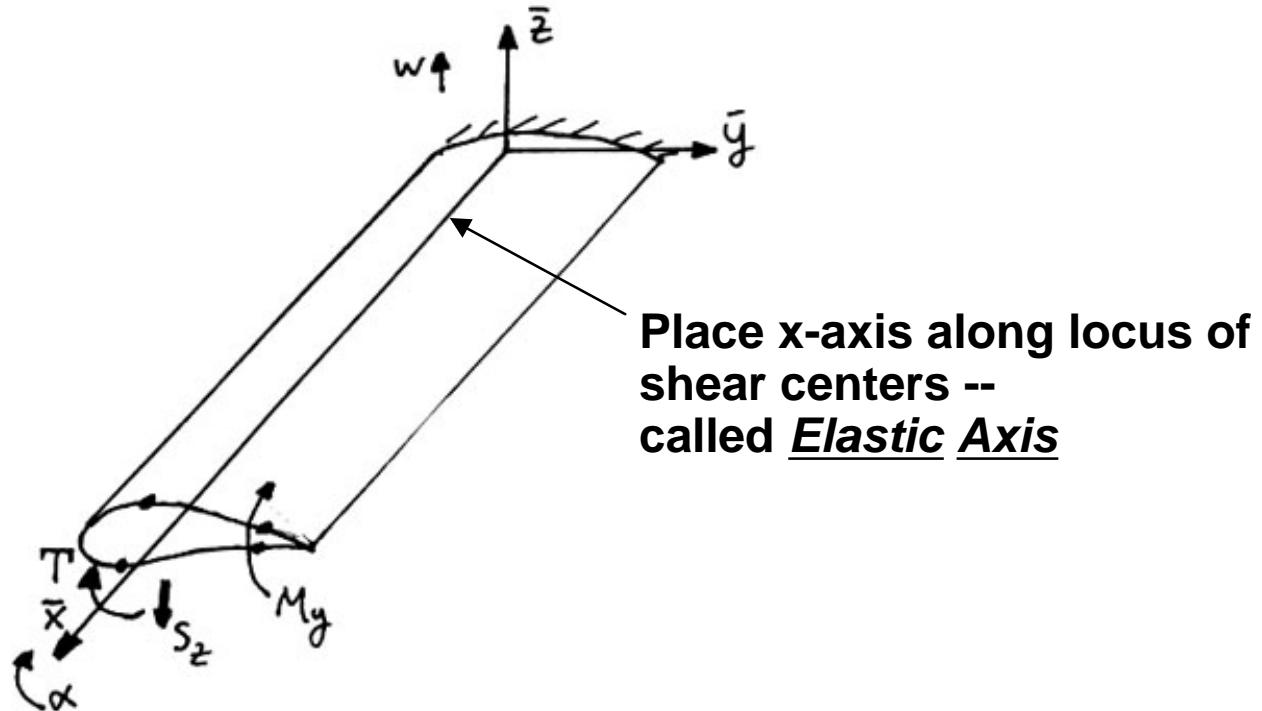
$$\frac{d\alpha}{dx} = \frac{T}{GJ}$$

$$\Rightarrow \boxed{J = \frac{2A}{\oint \frac{\bar{\bar{q}}}{t} ds}} \quad (\text{for closed section})$$

--> Summarizing, to find the deflections:

1. Obtain the section properties ( $EI$ ,  $GA$ ,  $GJ$ ) and the location of the shear center
2. Decompose load into moments, shears at shear center, and torque about shear center
3. Find (independently) bending, shearing, and twisting deflections about line of shear centers (elastic axis)
4. Sum deflections to obtain total deflection

**Figure 15.19 Representation of shell beam showing elastic axis**



Then, vertical deflection is:

$$w = \int w_B(\bar{x}) + \int w_S(\bar{x}) - \bar{y} \sin \alpha(\bar{x})$$

$\uparrow$                        $\uparrow$                        $\uparrow$   
 found from integrating appropriate  
 differential equation

Notes:

- Shear center important for aeroelastic analysis
- Shearing deflection ( $w_s$ ) appreciable for shell beams (often 10-20% of total)
- Generally
  - Stiff areas provide bending ( $EI$ )
  - Skin thickness provides torsion ( $GJ$ )

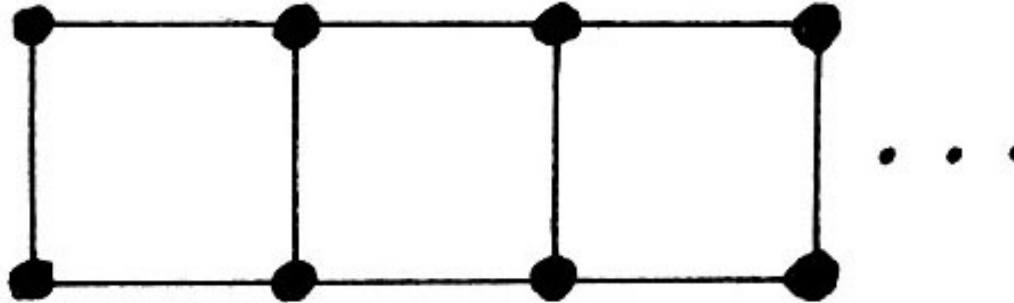
Have done this analysis for closed single-cell thin-skin shell beams. But can also apply this for other cases (3).

First consider...



# Multi-Cell Shell Beams

**Figure 15.20 Representation of general multi-cell shell cross-section**



**(looks like single-cell beam with additional webs)**

- > Bending stresses: calculated same way as before
- > Shear stresses: divide into two problems as before

(1) Pure Shear                      and                      (2) Pure Twist

Proceed as before:

1. Consider "Pure Shear" case
  - a) Apply joint equilibrium equation at each joint

Note: Still get  $(n - 1)$  equations but there are now  $(n + m - 1)$  variables

$n$  = number of joints

$m$  = number of cells

(extra  $q$ 's are in extra webs)

b) Apply Torque Boundary Condition

c) Apply No Twist Condition for **each cell**

(Note: no twist condition applies for any closed

$$\text{loop: } \oint \frac{q}{t} ds = 0)$$

This gives:

a)  $(n - 1)$  equations

b)  $(1)$  equation

c)  $(m)$  equations

---

$(n + m)$  equations

for  $(n + m)$  variables:

$(n + m - 1)$   $q$ 's

$(1)$   $d$

---

$(n + m)$  variables

## 2. Consider “Pure Torsion” case

- a) Apply joint equilibrium equation at each joint  
get:  $(n - 1)$  equations
- b) Apply Torque Boundary Condition



- c) Every cell must twist the same amount:

This is the **Equal Twist Condition**:

$$\left( \frac{d\alpha}{dx} \right)_j = \square C \quad \text{for each cell } j, \text{ this is a constant}$$

(True since assume shape of the cross-section is preserved)

This gives:

a)  $(n - 1)$  equations

b)  $(1)$  equation

c)  $(m - 1)$  equations

$(n + m - 1)$  equations

for  $(n + m - 1)$  q's

## 3. Sum results for “Pure Shear” and “Pure Torsion”





















