

MATH443
PARTIAL DIFFERENTIAL EQUATIONS
FIRST MIDTERM EXAM WITH SOLUTIONS

November 2, 2011 WEDNESDAY 10:40-12:30, SAZ01

QUESTIONS: Solve any four of the following five problems

[25]1. A Pfaffian differential equation

$$ydx - xdy + u(x, y, z)dz = 0$$

is integrable, where u is a differentiable function of x, y, z . **(a)** Find all possible integral surfaces of this equation. **(b)** Find the integral surface of this equation containing the curve $x = t^2$, $y = t$, $z = e^t$, $t \in I$.

Solution:(a) Integrability leads

$$X \cdot \text{curl}X = xu_x + yu_y - 2u = 0$$

where $X = (y, -x, u)$. Hence u satisfies a first order Partial Differential Equation. To find the integral surface of the Pfaffian differential equation we first solve $u(z, y, z)$. To solve it we use the Lagrange method.

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{0} = \frac{du}{2u}$$

We find that

$$F\left(\frac{u}{y^2}, \frac{x}{y}, z\right) = 0$$

or

$$u = y^2 f\left(\frac{x}{y}, z\right)$$

where f is an arbitrary function. With this solution the pfaffian DE becomes

$$ydx - xdy + y^2 f\left(\frac{x}{y}, z\right)dz = 0$$

or letting $\frac{x}{y} = \xi$ we get

$$d\xi + f(\xi, z)dz = 0$$

Let the integrating factor be $\phi(\xi, z)$. Then

$$\phi(d\xi + fdz) = d\psi$$

which means that $\psi(\xi, z) = c$ where c is an arbitrary constant and

$$\phi = \psi_\xi, \quad \phi f = \psi_z$$

or

$$f = \frac{\psi_z}{\psi_\xi}$$

Without losing generality we can let $\psi = h(\xi) - z$, then

$$f = -\frac{1}{h'(\xi)}$$

and the family of integral surfaces is

$$z = h\left(\frac{x}{y}\right)$$

where h is an arbitrary function. **(b)**. If an integral surface contains the curve $C : x = t^2, y = t, z = e^t$ we get

$$h(t) = e^t$$

Then the solution is

$$z = e^{\frac{x}{y}}$$

[25]**2. (a)** Find all possible surfaces orthogonal to the planes $x + 2y + cz = 1$, where c is an arbitrary real constant. **(b)** Find the surfaces orthogonal to the planes $x + 2y + cz = 1$ passing through the curve $z(x, x) = x^2$.

Solution: **(a)** Normal vector $(1, 2, c)$ of the plane is orthogonal to the surface S with normal $(p, q, -1)$. Then we get

$$p + 2q - c = 0$$

This is a linear first order pde. We solve it by using the Lagrange method. The solution is

$$z = cx + f(y - 2x)$$

where f is an arbitrary function. Hence the family surfaces (level surfaces) $z = cx + f(y - 2x)$ is orthogonal to the plane $x + 2y + cz = 1$. **(b)** Among this orthogonal family one contains the curve $C : x = t, y = t, z = t^2$. Hence

$$t^2 = ct + f(-t)$$

or

$$f(t) = t^2 + ct$$

then the solution is

$$z = cx + (y - 2x)^2 + c(y - 2x) = c(y - x) + (y - 2x)^2$$

[25]**3.** Verify that the equation

$$z(z + y^2)dx + z(z + x^2)dy - xy(x + y)dz = 0$$

is integrable and find its primitive.

Solution: **(a)** This pfaffian differential equation is integrable, i.e., $X = (z(z + x^2), z(z + y^2), -xy(x + y))$ and $X \cdot \text{curl}X = 0$. **(b)** To solve it we use the Natani's method. First let z be fixed. We find

$$\frac{x + y}{z - xy} = c(z) \tag{1}$$

Then let $x = 1$ we find

$$\frac{y(z + 1)}{z(y + 1)} = a \tag{2}$$

where a is a constant. Let $x = 1$ in (1) we get

$$\frac{1 + y}{z - y} = c(z) \tag{3}$$

Solving y from (2) we get $y = \frac{zc-1}{1+c}$ and inserting it in (3) we get

$$c(z) = \frac{1}{a-1} \frac{1}{z}$$

Hence the solution (the primitive) is

$$F(x, y, z) = \frac{z(x + y)}{z - xy} = b$$

where $b = \frac{1}{a-1}$ is an arbitrary constant

[25]4. Find the integral surface of the equation

$$(x + y - z)p - (y + x + z)q = 2z. \quad (4)$$

passing through the curve $z(x, x) = 1$.

Solution: We use the Lagrange method to solve this problem

$$\frac{dx}{x + y - z} = -\frac{dy}{x + y + z} = \frac{dz}{2z}$$

which has solutions

$$u = x + y + z = c_1, \quad (5)$$

$$v = y + \frac{x + y + z}{2} \ln z = c_2 \quad (6)$$

The integral surface contains the curve $C : x = t, y - t, z = 1$, then we find

$$2c_2 - c_1 + 1 = 0$$

or

$$y - x - z + 1 + (x + y + z) \ln z = -0$$

[25]5. Find a complete integral of the equation $p^2 y(1 + x^2) - q x^2 = 0$.

Solution: We solve such problems first finding a solution of the set of equations

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z}$$

Where $f = p^2 y(1 + x^2) - q x^2 = 0$. Using first and fourth terms we get $p^2 = a^2 \frac{x^2}{1+x^2}$ and $q = ay$. Then

$$dz = \frac{\sqrt{a} x}{\sqrt{1+x^2}} dx + ay dy$$

The solution is

$$z = \sqrt{a} \sqrt{1+x^2} + \frac{a}{2} y^2 + b$$

where $a \geq 0$ and b are arbitrary constants. If we consider the second and fifth terms we get the same solutions again.