

## Complex form of Fourier series and Fourier integral

### Fourier integral:

Fourier integral is an extension of Fourier series in non-periodic functions. Here integration is used instead of summation in a Fourier series.

Let us consider a function  $f(x)$ . Let us consider the following conditions for the Fourier integral of the given function  $f(x)$

- I. The function  $f(x)$  and its derivative  $f'(x)$  both are piecewise continuous in every finite interval.
- II.  $\int_{-\infty}^{\infty} |f(x)| dx$  Converges i.e.  $f(x)$  is absolutely integrable in  $(-\infty, \infty)$ . The Fourier series of  $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$  can be expressed in an integral form as described in the following section.

The Fourier series expression of  $f(x)$  is given by

$f(x) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$   $(-L < x < L)$  can be expressed in Fourier integral.

$$\text{i.e. } f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

$$\text{Where } A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv \quad \& \quad B(w) = \int_{-\infty}^{\infty} f(v) \sin wv dv$$

This representation is known as the Fourier integral for the function  $f(x)$ .

**NOTE:** If  $f(x)$  is not continuous at  $x=a$  then  $\frac{f(a+0) + f(a-0)}{2}$

- where  $f(a+0) = \lim_{h \rightarrow 0} f(a+h)$  and  $f(a-0) = \lim_{h \rightarrow 0} f(a-h)$
- $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$
- $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$

**Question:** Find the Fourier integral representation of the function

$$f(x) = \begin{cases} \cos x & , |x| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

**Solution:**

We know that  $f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin x]dw$ -----(1)

Where,  $A(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v)\cos wv dv$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos v \sin v dv$$

Now,

$$A(w) =$$

$$\frac{2}{\pi} \int_0^{\pi/2} 2\cos v \sin wv dv = \frac{1}{\pi} \int_0^{\pi/2} [\cos(1+w)v + \cos(w-1)v]dv$$

$$= \frac{1}{\pi} \left( \frac{\sin(1+w)v}{1+w} + \frac{\sin(w-1)v}{w-1} \right)_0^{\pi/2}$$

$$= \frac{1}{\pi} \left( \frac{\sin(1+w)\pi/2}{1+w} + \frac{\sin(w-1)\pi/2}{w-1} \right)$$

$$= \frac{1}{\pi(w^2-1)} [w(\sin(1+w)\frac{\pi}{2} + \sin(w-1)\frac{\pi}{2} - \sin(w+1)\frac{\pi}{2}]$$

$$= \frac{1}{\pi(w^2-1)} [2w \sin w \frac{\pi}{2} \cos \frac{\pi}{2} - 2\cos w \frac{\pi}{2} \sin \frac{\pi}{2}]$$

$$= \frac{2}{\pi(w^2-1)} \left[ \cos \left( w \frac{\pi}{2} \right) \right]$$

From equation (1), we get

$$f(x) = \int_0^\infty \frac{2}{\pi(w^2-1)} \cos \left( w \frac{\pi}{2} \right) \cos wx dw$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos \left( w \frac{\pi}{2} \right) \cos wx}{1-w^2} dw, \text{ Provided } w \neq 1$$

### Fourier Integral Theorem:

If a function  $f(x)$  (i) is piecewise continuous in every finite interval  $a \leq x \leq b$

i.e.  $f(t-0) = f(t+0) = f(t)$  i.e.  $\lim_{x \rightarrow t-0} f(x) = \lim_{x \rightarrow t+0} f(x) = f(t)$

ii. Has left hand derivative  $\lim_{h \rightarrow 0^-} \frac{f(x+h)-f(x)}{h}$  and a right hand derivative

$\lim_{h \rightarrow 0^+} \frac{f(x+h)-f(x)}{h}$  at every point and.

iii. The integral  $\lim_{a \rightarrow -\infty} \int_a^0 |f(x)| dx + \lim_{b \rightarrow -\infty} \int_0^b |f(x)| dx$  exists then  $f(x)$  represented by a Fourier integral  $f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx] dw$

Where  $A(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v)\cos wv dv$ ,  $B(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v)\sin wv dv$

**Note:** If the  $f(x)$  is discontinuous at a point  $x = \alpha$  (i.e.  $f(\alpha - 0)$ ,  $f(\alpha + 0)$  and  $f(\alpha)$  are not all equal), then the value of the Fourier integral equals the average of the left and right hand limits of  $f(x)$  at  $x = \alpha$

$$\frac{1}{2} [f(\alpha - 0) + f(\alpha + 0)]$$

### Equivalent forms of Fourier integral theorem:

Fourier integral theorem for the function  $f(x)$  can also be rewritten in the following forms then  $f(x)$  is continuous.

- i.  $f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(u) \cos \alpha (x - u) du d \alpha$
- ii.  $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(u) e^{i\alpha(x-u)} du d \alpha$
- iii.  $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\alpha x} d \alpha \int_{-\infty}^\infty f(u) e^{-i\alpha u} du$

**Example:** Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 0 & , x < 0 \\ \pi e^{-x} & , x > 0 \end{cases} \quad \text{Hence evaluate } \int_0^\infty \frac{\cos xw + w \sin xw}{1+w^2} dw.$$

**Solution:**

We know  $f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx] dw$

Where,  $A(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v)\cos wv dv$   $B(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v)\sin wv dv$

Here,  $A(w) = \frac{1}{\pi} \int_{-\infty}^\infty \pi e^{-v} \cos wv dv = \int_0^\infty e^{-v} \cos wv dv$

$$B(w) = \frac{1}{\pi} \int_0^\infty \pi e^{-v} \sin wv dv = \int_0^\infty e^{-v} \sin wv dv$$

$$A(w) + iB(w) = \int_0^\infty e^{-v} (\cos v + i \sin wv) dv$$

$$\begin{aligned}
&= \int_0^{\infty} e^{-v} e^{iwv} dv = \int_0^{\infty} e^{(iw-1)v} dv = \left. \frac{e^{(iw-1)v}}{iw-1} \right]_0^{\infty} \\
&= \frac{-1}{iw-1} = \frac{1}{1+w^2} + i \frac{w}{1+w^2}
\end{aligned}$$

Therefore,  $A(w) = \frac{1}{1+w^2}$ ,  $B(w) = \frac{w}{1+w^2}$  and  $f(x) = \int_0^{\infty} \frac{\cos xw + w \sin xw}{1+w^2} dw$

The function  $f(x)$  is discontinuous at  $x=0$ . Therefore the value of

$$f(0) = \frac{1}{2} \left[ \lim_{x \rightarrow 0-0} f(x) + \lim_{x \rightarrow 0+0} f(x) \right] = 1/2 (0 + \pi e^{-0}) = \frac{\pi}{2}$$

We know  $f(x) = 0$  if  $x < 0$  and  $f(x) = \pi e^{-x}$  in  $x > 0$

Therefore

$$\int_0^{\infty} \frac{\cos xw + w \sin xw}{1+w^2} dw = \begin{cases} 0, & \text{if } x > 0 \\ \frac{\pi}{2}, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases}$$