

## Fourier Integral

### Fourier Sine and Cosine Integrals:

The Fourier integral becomes simpler when the given function is either an even or odd function. The Fourier integral of  $f(x)$  is described as

$$f(x) = \int_0^{\infty} A(w) \cos wx + B(w) \sin wx dw$$

$$\text{Where } A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv \quad \text{and} \quad B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv$$

#### Case-1:

Suppose  $f(x)$  is an even function i.e.  $f(-x)=f(x)$ . Now  $\cos wx$  is also an even function and  $\sin wx$  is an odd function. Therefore  $f(x) \cos wx$  is also an even function and  $f(x) \sin wx$  is an odd function.

$$\text{Now } A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv = \frac{2}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv$$

$$\text{And } B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv = 0$$

$$\text{Therefore } f(x) = \int_0^{\infty} A(w) \cos wx dw$$

This is known as the Fourier Cosine integral representation.

#### Case-2:

Suppose  $f(x)$  is an odd function i.e.  $f(-x)=-f(x)$ . Now  $\sin wx$  is also an odd function then  $f(x) \cos(wx)$  is also an odd function and  $f(x) \sin wx$  is an even function.

$$\text{Now} \quad A(w) =$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv = 0 \text{ and}$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv = \frac{2}{\pi} \int_0^{\infty} f(v) \sin wv dv$$

$$\text{Therefore } f(x) = \int_0^{\infty} B(w) \sin wx dw$$

This is known as the Fourier sine integral.

### EXERCISE 2.4

1. Show that  $\int_0^{\infty} \frac{\cos\left(\frac{\pi w}{2}\right) \cos(xw)}{(1-w)^2} dw = \begin{cases} \frac{\pi}{2} \cos x & , |x| < \frac{\pi}{2} \\ 0 & , |x| > \frac{\pi}{2} \end{cases}$
2. Find the Fourier cosine integral representation of the function  $f(x) = \begin{cases} x & , 0 < x < a \\ 0 & , x > a \end{cases}$
3. Find the Fourier sine integral representation of the function
  - a.  $f(x) = \begin{cases} e^x & , 0 < x < 1 \\ 0 & , x > 1 \end{cases}$
  - b.  $f(x) = \begin{cases} \sin x & , 0 \leq x \leq \pi \\ 0 & , x > \pi \end{cases} \sin x$
  - c.  $f(x) = \begin{cases} \pi - x & , 0 < x < \pi \\ 0 & , x > \pi \end{cases}$
4. Prove that  $\int_0^{\infty} \frac{w^3 \cos xw}{w^4+4} dw = \frac{\pi}{2} e^{-x} , x > 0$

**Properties of Fourier integral:**

1.  $f(ax) = \frac{1}{a} \int_0^{\infty} A\left(\frac{w}{a}\right) \cos wx dw \quad a > 0$
2.  $xf(x) = \int_0^{\infty} B^*(w) \sin wx dw$  Where
 
$$B^* = -\frac{dA}{dw} \quad \text{and} \quad A(w) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos wv dv$$
3.  $x^2 f(x) = \int_0^{\infty} A^*(w) \cos wx dw , \quad A^* = -\frac{d^2 A}{dw^2}$