

### Fourier Sine Transform:

Let  $f(x)$  be defined for  $0 \leq x < \infty$  and let  $f(x)$  be extended as an odd function in  $(-\infty, \infty)$  satisfying the condition of Fourier integral theorem. If at the point of

continuity  $\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin wt \, dt$

Then sine transform of  $\hat{f}_s(w)$  is represented by the inverse Fourier sine transformation  $f(x)$ .

Inverse Fourier sine transformation of  $f(x)$ .

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(w) \sin wx \, dw$$

### Fourier Cosine Transform:

If  $f(x)$  be defined for  $0 \leq x < \infty$  and let  $f(x)$  be extended as an even function in  $(-\infty, \infty)$  satisfying the condition of Fourier integral theorem. If at the point of

continuity  $\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos wt \, dt$

Then the cosine transform of  $\hat{f}_c(w)$  is given by the function

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_c(w) \cos wx \, dw$$

Here  $\hat{f}_c(w)$  is called Fourier cosine transform of  $f(x)$  and  $f(x)$  is called the inverse Fourier cosine transform of  $\hat{f}_c(w)$ .

### Properties of Sine and Cosine Transform:

- **Linearity Property:**

Let  $a, b$  be any constant and  $f(x), g(x)$  be the any function of  $x$ . then Fourier transform of  $f(x)$  and  $g(x)$  exist. Such that

$$\mathcal{F}(af + bg) = a \mathcal{F}(f) + b \mathcal{F}(g)$$

- **Derivative Property:**

Let  $f(x)$  be continuous on the  $x$ -axis and  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Furthermore it is absolutely integrable on the  $x$ -axis. Then  $f'(x)$  is piecewise continuous on each finite interval.

$$a. \mathcal{F}\{f'(x)\} = w \mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$$

$$b. \mathcal{F}_s\{f'(x)\} = -w \mathcal{F}_c\{f(x)\}$$

$$c. \mathcal{F}_c\{f''(x)\} = -w^2 \mathcal{F}_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$d. \mathcal{F}\{f''(x)\} = -w^2 \mathcal{F}\{f(x)\} + \sqrt{\frac{2}{\pi}} w f(0)$$

**Note:** If  $f$  and its 1<sup>st</sup> (n-1) derivatives are continuous and its  $n$ th derivative is piecewise smooth in

$(-\infty, \infty)$ , then  $\mathcal{F}\{f^n(t)\} = (-iw)^n \mathcal{F}\{f(x)\}$  provided that  $f$  and its derivatives are absolutely integrable. In addition we assume that  $f$  and its 1<sup>st</sup> (n-1) derivatives approaches to zero as  $t \rightarrow \infty$

### Convolution Theorem:

Suppose that  $f(x)$  and  $g(x)$  are piecewise continuous, bounded and absolutely integrable on the  $x$ -axis. Then

$$\mathcal{F}(f * g) = \sqrt{2\pi} \mathcal{F}(f) \mathcal{F}(g)$$

**Question:** Find the Fourier transform of  $xe^{-x^2}$

**Solution:**

Here  $f(x) = e^{-x^2}$  then  $f'(x) = -2xe^{-x^2} \Rightarrow \frac{-1}{2} f'(x) = xe^{-x^2}$

$\therefore \mathcal{F}(xe^{-x^2}) = \frac{-1}{2} \{\mathcal{F} f'(x)\} = \frac{-1}{2} iw \mathcal{F}(e^{-x^2})$  ( $\because$  derivative property)

$$\text{But } \mathcal{F}(e^{-ax^2}) = \frac{e^{-w^2/4a}}{\sqrt{2a}}$$

$$\text{i.e. } \mathcal{F}(xe^{-x^2}) = -\frac{iw}{2} \frac{e^{-w^2/4}}{\sqrt{2}} = -\frac{iw}{2\sqrt{2}} e^{-w^2/4}$$

**Question:** Find the Fourier cosine transform of

$$f(x) = \begin{cases} 1 & , \text{ if } 0 < x < 1 \\ -1 & , \text{ if } 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

**Answer:**

We know that Fourier cosine transform is given by

$$\begin{aligned} \hat{f}_s(w) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[ \int_0^1 f(x) \cos wx \, dx + \int_1^2 f(x) \cos wx \, dx + \int_2^{\infty} f(x) \cos wx \, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left[ \int_0^1 \cos wx \, dx - \int_1^2 \cos wx \, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left\{ \left[ \frac{\sin wx}{w} \right]_0^1 - \left[ \frac{\sin wx}{w} \right]_1^2 \right\} \\ &= \sqrt{\frac{2}{\pi}} \left[ \frac{\sin w}{w} - 0 - \frac{1}{w} (\sin 2w - \sin w) \right] \\ &= \sqrt{\frac{2}{\pi}} \left[ \frac{2 \sin w - \sin 2w}{w} \right] \end{aligned}$$

**Question:** Find the Fourier sine transform of  $f(x) = e^{-x}$

**Solution:**

By definition of Fourier sine transform, we get

$$\begin{aligned} \hat{f}_s(w) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin wx \, dx \\ &= \sqrt{\frac{2}{\pi}} \left[ \frac{e^{-x}}{1+w^2} (-\sin wx - w \cos wx) \right]_0^{\infty} = \sqrt{\frac{2}{\pi}} \left( \frac{w}{1+w^2} \right) \end{aligned}$$

And the inverse Fourier transform of  $f(x)$  is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}_s(w) \sin wx \, dx = \frac{2}{\pi} \int_0^{\infty} \frac{w}{1+w^2} \sin wx \, dw$$

**Question:** Find the Fourier transform of  $f(x) = \begin{cases} e^{-ikx}, & a < x < b \\ 0, & a < x \text{ and } x > b \end{cases}$

**Solution:**

By definition of Fourier transform, we get-

$$\begin{aligned} f(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^a f(x) e^{-iwx} dx + \int_a^b f(x) e^{-iwx} dx + \int_b^{\infty} f(x) e^{-iwx} dx \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-ikx} e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-i(k+w)x} dx = \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{-i(k+w)x}}{-i(k+w)} \right]_a^b \\ &= \frac{1}{\sqrt{2\pi}} \frac{i}{k+w} \left[ e^{-(k+w)b} - e^{-i(k+w)a} \right] = \frac{-i}{\sqrt{2\pi}} \left[ \frac{e^{-i(k+w)b} - e^{-i(k+w)a}}{k+w} \right] \end{aligned}$$

### EXERCISE 2.5

1. Find the Fourier transform of the following

a.  $f(x) = \begin{cases} e^x, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

2.  $f(x) = \begin{cases} k, & a < x < b \\ -k, & b < x < c \\ 0, & \text{Otherwise} \end{cases}$

3. Find the Fourier Cosine transform of the following

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & \text{Otherwise} \end{cases}$$

a.  $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & \text{Otherwise} \end{cases}$

4. Find the Fourier Cosine transform of the following

a.  $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

b.  $f(x) = e^{-x}, x \geq 0$