

Fourier series expansion of period 2L

Some Important Integration Results:-

$$* \int_{-\pi}^{\pi} \sin nx \, dx = \int_{-\pi}^{\pi} \cos nx \, dx = \int_0^{2\pi} \sin nx \, dx = \int_0^{2\pi} \cos nx \, dx = 0$$

$$* \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} \pi, & \text{for } m = n \\ 0, & \text{for } m \neq n \end{cases}$$

$$* \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 = \int_0^{2\pi} \sin mx \cos nx \, dx$$

$$* \int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 = \int_0^{2\pi} \cos mx \sin nx \, dx$$

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Example – 1. Expand in Fourier series $f(x) = x + x^2$, $(-\pi < x < \pi)$,

Hence prove that $1 + \frac{1}{2^2} + \dots + \frac{1}{n^2} = \frac{\pi^2}{6}$

Ans:-The Fourier Series expansion of $f(x)$ with period 2π in the interval $-\pi < x < \pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1)$$

Now

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \, dx = \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cdot \cos nx \, dx$$

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$$\begin{aligned}
&= \frac{1}{\pi} \left\{ \left[(x + x^2) \left(\frac{\sin nx}{n} \right) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (1 + 2x) \frac{\sin nx}{n} \cdot dx \right\}, \text{ since } \left[(x + x^2) \left(\frac{\sin nx}{n} \right) \right]_{-\pi}^{\pi} = 0 \\
&= \frac{1}{\pi} \left\{ \left[\left(\frac{1+2x}{n} \right) \left(\frac{-\cos nx}{n} \right) \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2}{n} \left(\frac{-\cos nx}{n} \right) dx \right\} \\
&= \frac{1}{\pi} \left\{ \left[\left(\frac{1+2x}{n} \right) \left(\frac{\cos nx}{n} \right) \right]_{-\pi}^{\pi} - \left(\frac{2}{n} \left(\frac{\sin nx}{n} \right) \right)_{-\pi}^{\pi} \right\}, \text{ since } \left(\frac{2}{n} \left(\frac{\sin nx}{n} \right) \right)_{-\pi}^{\pi} = 0 \\
&= \frac{-1}{n^2 \pi} [\{(1 + 2\pi) \cos n\pi - (1 - 2\pi) \cos n\pi\} + 0] \\
&= \frac{4}{n^2} \cos n\pi = \frac{4}{n^2} (-1)^n
\end{aligned}$$

Therefore
$$a_n = \begin{cases} \frac{4}{n^2} & \text{if } n \text{ is even} \\ \frac{-4}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

$$a_1 = \frac{-4}{1^2}, \quad a_2 = \frac{4}{2^2}, \quad a_3 = \frac{-4}{3^2}, \quad a_4 = \frac{4}{4^2} \dots \text{and so on.}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2) \cdot \sin nx \, dx \\
&= \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} x \sin nx \, dx + \int_{-\pi}^{\pi} x^2 \sin nx \, dx \right\} \\
&\quad (x^2 \sin nx \, dx = \text{odd function} = 0) \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{1}{\pi} [x(-\cos nx/n) + \sin nx/n^2]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} (-2\pi \cos n\pi/n) = \frac{-2 \cos n\pi}{n} = \frac{-2}{n} (-1)^n
\end{aligned}$$

Therefore
$$b_n = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd} \\ \frac{-2}{n} & \text{if } n \text{ is even} \end{cases}$$

Put the value of a_0, a_n, b_n in equation (1), we get

$$f(x) = \frac{2\pi^3}{6} + \left(-\frac{4}{12} \cos x + \frac{4}{2^2} \cos 2x - \frac{4}{3^2} \cos 3x + \dots \right) + \left(\frac{2}{1} \sin x - \frac{2}{2} \sin 2x + \frac{2}{3} \sin 3x - \dots \right)$$

$$= \frac{2\pi^3}{6} - 4 \left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right) + 2 \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \dots \right)$$

Put $x=\pi$ then $\frac{1}{2}[f(-\pi + 0) + f(\pi - 0)] = \frac{1}{2}[-\pi + \pi^2 + \pi + \pi^2] = \pi^2$

$$\pi^2 = \frac{\pi^2}{3} + 4 \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right) + 2 * 0$$

$$= \pi^2 - \frac{\pi^2}{3} = 4 \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right)$$

$$\frac{2\pi^2}{3*4} = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right) \Rightarrow \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right) = \pi^2/6$$

EXERCISE 2.1

1. Find the smallest positive period of the following function.

(a) $\sin \frac{2\pi x}{k}$, $\cos \frac{2\pi n x}{k}$ (b) $\tan \frac{2n\pi x}{k}$, $\cot \frac{n\pi x}{k}$

2. Find the Fourier series expansion of the following functions which is assumed to have the period 2π .

(a) $f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$, and hence deduce $\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right) = \frac{\pi}{4}$

(b) $f(x) = x^2 (-\pi < x < \pi)$, hence show that $\frac{\pi^2}{6} = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$

(c) $f(x) = \frac{x^2}{2} (-\pi < x < \pi)$, and hence show that

$$\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right) = \frac{\pi^2}{6} \text{ and } \left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right) = \frac{\pi^2}{12}$$

(d) $f(x) = \begin{cases} -\pi, & \pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$, and deduce that $\left(1 + \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right) = \frac{\pi^2}{8}$

3. Find the Fourier series expansion of $f(x)$ of the $(-\pi < x < \pi)$

$$(a) f(x) = \begin{cases} x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases} (b) f(x) = x + x^2, (-\pi < x < \pi), \text{ hence to show that } \\ \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \right) = \frac{\pi^2}{12}$$

Fourier series in an Arbitrary (period) Interval:-

Let $f(x)$ be the periodic function with period 2π consider as arbitrary period $2L$ then the Fourier series expansion with period $2L$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), (-L < X < L)$$

Where a_0, a_n, b_n are *fourier Coeffieients* .

Here

$$\left. \begin{aligned} a_0 &= 1/L \int_{-\pi}^{\pi} f(x).dx \\ a_n &= 1/L \int_{-\pi}^{\pi} f(x). \cos \frac{n\pi x}{L} dx \\ b_n &= 1/L \int_{-\pi}^{\pi} f(x). \sin \frac{n\pi x}{L} dx \end{aligned} \right\} \text{Euler's form of Fourier coefficient with arbitrary period}$$

$2L$

Example – 2. Find the Fourier series for $f(x) = 1 - x^2$ in $(-1 < x < 1)$

The Fourier series expansion of $f(x)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \text{ Here } 2L = 2 \Rightarrow L = 1$$

$$\text{Now } a_0 = \frac{1}{L} \int_{-\pi}^{\pi} f(x). dx = \frac{1}{1} \int_{-1}^1 (1 - x^2) dx \Rightarrow 2 \int_0^1 (1 - x^2) dx = 2 \left(x - \frac{x^3}{3} \right)_0^1$$

$$= 4/3$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x). \cos \frac{n\pi x}{L} dx$$

$$\begin{aligned}
&= \frac{1}{1} \int_{-1}^1 (1 - x^2) \cos n\pi x \, dx = 2 \int_0^1 (1 - x^2) \cos n\pi x \, dx \\
&= 2 \left\{ \left[(1 - x^2) \left(\frac{\sin n\pi x}{n\pi} \right) \right]_0^1 + \int_0^1 (2x \sin n\pi x / n\pi) \, dx \right\} \\
&= 2 \left[(1 - x^2) \frac{\sin(n\pi x)}{n\pi} - \frac{2x \cos n\pi x}{n^2 \pi^2} + \frac{2}{n^3 \pi^3} \sin(n\pi x) \right]_0^1 = \left[-\frac{4\pi}{n^2 \pi^2} \cos n\pi x \right]_0^1 = \\
&= -\frac{4}{n^2 \pi^2} (-1)^n = \frac{4}{n^2 \pi^2} (-1)^{n+1}
\end{aligned}$$

Now $b_n = \frac{1}{L} \int_{-L}^L f(x) \cdot \sin \frac{n\pi x}{L} \, dx$

$$= \frac{1}{1} \int_{-1}^1 (1 - x^2) \sin \frac{n\pi x}{1} \, dx = 0 \quad (\text{as the product is an odd function}).$$

Therefore the required series is

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} \right) = \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^{n+1} \quad (\text{or}) \quad \frac{2}{3} + \\
&\frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (-1)^{n-1}
\end{aligned}$$

Exercise 2.2

1. Find the Fourier series expansion of the following function which are assumed to the arbitrary period $2L$.

(a) $f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ -1 & \text{if } 0 < x < 1 \end{cases}$

(b) $f(x) = |x|, (-2 < x < 2)$

(c) $f(x) = 3x^2, (-1 < x < 1)$

(d) $f(x) = \pi \sin \pi x, (0 < x < 1)$

(e) $f(x) = \frac{\pi x^3}{2}, (-1 < x < 1)$