

# Fourier series expansion of periodic function

## Introduction:-

The French physicist and mathematician Jean-Baptiste Joseph Fourier (1768-1830) first solved the heat conduction problem which can be formulated by differential equation with boundary conditions, using an infinite series of sine and cosine functions.

Fourier series plays an important role in designing and analysing electrical and electronic communication systems. In engineering systems, inputs are obtained in the form of electrical signals, electromagnetic signals, magnetic waves, radiation, sound waves, vibrations etc. The Fourier coefficients are features for the analysis of signals and images as well as for solving the problems in pattern recognition.

In this chapter, basis of Fourier Series, determination of Fourier coefficients, Fourier Series in an arbitrary interval, Fourier Series for even and odd functions, Half range Fourier Series are discussed.

## Basis of Fourier series:-

The Fourier Series of a function is an infinite series of sine and cosine functions, so the basis of Fourier Series are developed based on the principle of periodic functions, trigonometric series and Dirichlet's conditions.

## Periodic Functions:-

A function  $f(x)$  is called periodic if  $f(x+p) = f(x)$  for all real  $x$  and some positive  $p$ , where  $p$  is called the period of  $f(x)$ .

In trigonometry, sine and cosine functions are periodic functions, we exploit these sine and cosine functions for the development of Fourier series.

**Example:-** We know that  $\sin(2n\pi+x) = \sin x$  and  $\cos(2n\pi+x) = \cos x$ , Here  $2\pi$  is the period and  $n$  is an integer ( $n \in \mathbb{Z}$ )

## Fundamental Period:-

The fundamental period of a periodic function  $f(x)$  is its smallest positive period.

- If  $f(x)$  is periodic function with period  $p$ , then the function  $f(ax+b)$  (where  $a>0$ ) is periodic with period  $p/a$ , i.e.  $p/a$  is the smallest positive period of  $f(ax+b)$ .
- Fundamental period of a function  $f(x)=\sin x$  or  $f(x)=\cos x$  is  $2\pi$ (which are also called sinusoidal periodic function) whereas  $f(x)=\tan x$  or  $f(x)=\cot x$  periodic with fundamental period  $\pi$ .

### Properties of Periodic Functions:-

- If  $p$  is the fundamental period of a periodic function  $f(x)$ , then  $np$  is called a period of  $f(x)$  for an integer  $n\neq 0$  i.e.  $f(x)=f(x+np)$

**Example:-**  $\sin(2n\pi+x) = \sin x, \forall n \in \mathbb{Z}$  and here  $p=2\pi$

- If  $f(x)$  and  $g(x)$  are periodic functions with period  $p$ , then  $h(x)=af(x)+bg(x)$  is also a periodic functions with period  $p$  for some constants  $a$  and  $b$ .

**Example:-** if  $f(x) = \sin x, g(x) = \cos(x)$  then  $h(x)=2\sin x+3\cos x$

i.e.  $h(x)=2\sin(x+2n\pi)+3\cos(x+2n\pi)$

$=2\sin x+3\cos x$ , where  $p=2\pi$ .

- If  $f(x)$  is a periodic function with period  $p$ , then  $f(ax)$  is also a periodic function with period  $p/a$ , where 'a' is a constant and  $a\neq 0$ .

**Example:-**  $f(x)=\sin(2n\pi x/k)$

Comparing with the  $\sin ax$ , here  $a=2n\pi/k$  and  $\sin x$  is periodic with period  $p=2\pi$

$$\therefore \frac{p}{a} = \frac{2\pi}{\frac{2n\pi}{k}} = \frac{k}{n}$$

- The period of sum of periodic functions with different period is the LCM(least common multiple) of these periods.
- If  $f(x)=c$ , where  $c$  is a constant, then period of  $f(x)$  is any positive value of  $p$ .

### Trigonometric Series:-

A Series with trigonometric functions such as  $\sin x, \cos x, \tan x, \cot x, \sec x, \operatorname{cosec} x$  is called a trigonometric series.

Ex:-

$$1. \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, |x| < 1$$

$$2. \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, |x| < 1$$

$$3. \tan x = x + \frac{x^3}{3} + \frac{2x^5}{4!} + \dots$$

### Euler's Formulas for the Fourier Coefficients:-

If  $f(x)$  is periodic function with period  $2\pi$ . Then the Fourier series expansion of  $f(x)$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where  $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x). dx$  or  $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x). dx$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x). \cos nx dx \text{ or } \frac{1}{\pi} \int_{-\pi}^{\pi} f(x). \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x). \sin nx dx \text{ or } \frac{1}{\pi} \int_{-\pi}^{\pi} f(x). \sin nx dx$$

Here  $a_0, a_n, b_n$  are called Euler formula for the Fourier Series. Here  $[a_n]$  and  $[b_n]$  are converges to a continuous function  $f(x)$  on the interval  $[-\pi, \pi]$ .