

Half Range Expansion of a Fourier series:-

Even and odd functions:-

Even Function:-

A function $f(x)$ is called even if $f(-x) = f(x)$ for all value of x .

Properties of even functions:-

- The graph of an even function is always symmetrized about the y-axis.
- $f(x)$ contains only even power of x and may contains only $\cos x, \sec x$.
- Sum of two even function is even.
- Product of two even function is even.
- $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

Odd Function:-

A function $f(x)$ is called odd if $f(-x) = -f(x)$, $\forall x \in \mathbb{R}$

- The graph of an odd function is always symmetric about the origin and lie in opposite quadrant (1st and 3rd).
- $f(x)$ contains only odd power of x and may contains only $\sin x, \operatorname{cosec} x$.
- $\int_{-a}^a f(x) dx = 0$.
- Sum of two odd function is odd function .
- Product of an odd function and even function is an odd function.
- Product of two odd function is an even function .

Neither Even nor Odd functions:-

A function $f(x)$ which is not satisfying the even and odd condition is called neither even nor odd function.

Some Useful Formulas:-

- $\sin n\pi = \cos (n+1/2) \pi = 0 \forall n \in \mathbb{Z}$

i.e. $\sin \pi = \sin 2\pi = \sin (3\pi) = \sin (-\pi) = \sin(-2\pi) = \dots = 0$

i.e. $\cos \pi/2 = \cos 3\pi/2 = \cos 5\pi/2 = \dots = 0$

- $\cos n\pi = \sin(n+1/2) \pi = (-1)^n \forall n \in \mathbb{Z}$

$$\cos \pi = \cos 3\pi = \cos 5\pi = \dots = -1$$

$$\sin \pi/2 = \sin 5\pi/2 = \sin 9\pi/2 = \dots = 1$$

$$\cos 0 = \cos 2\pi = \cos 4\pi = \dots = 1$$

$$\sin 3\pi/2 = \sin 7\pi/2 = \sin 11\pi/2 = \dots = -1$$

- If $x=x_0$ is the part of the finite discontinuity then sum of the fourier series

$$f(x) = \frac{1}{2} \left[\lim_{h \rightarrow 0} f(x_0 - h) + \lim_{h \rightarrow 0} f(x_0 + h) \right]$$

$$f(x) = 1/2 [f(x_0 - 0) + f(x_0 + 0)]$$

Even and odd Extensions of a Function:-

Let f be a function given on the interval $(0,a)$, we define the even/odd extension of f to be even/odd functions on the interval $(-a,a)$, which coincides with f on the half interval $(0,a)$.

Even Extension:-

$$f_E = \text{even } f = \begin{cases} f(-x), & -a < x < 0 \\ f(x), & 0 < x < a \end{cases}$$

Odd Extension:-

$$f_O = \text{odd } f = \begin{cases} -f(-x), & -a < x < 0 \\ f(x), & 0 < x < a \end{cases}$$

Half Range Expansion of a Fourier series:-

Suppose a function is defined in the range $(0,L)$, instead of the full range $(-L,L)$. Then the expansion $f(x)$ contains in a series of sine or cosine terms only. The series is termed as half range sine series or half range cosine series.

- If $f(x)$ is taken to be an odd function, its Fourier series expansion will consists of only sine terms. Hence the Fourier series expansion of $f(x)$ represents "Half range expansion of Fourier sine series".

i.e $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$, ($0 < x < L$), where $b_n = \frac{2}{L} \int_0^L f(x) \sin \left(\frac{n\pi x}{L} \right) dx$.

- If $f(x)$ is taken to be an even function, then its Fourier series expansion will consist of constant and cosine terms. Therefore the Fourier series expansion of $f(x)$ represents “Half range expansion of Fourier series” i.e.

$$f(x) = f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} \right)$$

where $a_0 = \frac{2}{L} \int_0^L f(x) dx$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \text{ for all } (n \geq 1) n \in \mathbb{Z}$$

Here

a_0, a_n, b_n are called coefficients half range Fourier cosine series.

Example – 3:- Determine the half range Fourier series sine series for

$$f(x) = e^x \text{ in } 0 < x < 1$$

Solution:-

Given that $f(x) = e^x$ $0 < x < 1$ we know that half range Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi x}{L} \right) \text{ here } l = 1$$

$$\text{Now } b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= 2 \int_0^1 e^x \sin n\pi x dx$$

$$= 2 \left[\frac{e^x}{1+n^2\pi^2} (\sin n\pi x - n\pi \cos n\pi x) \right]_0^1 = \left[\frac{2n\pi}{1+n^2\pi^2} (-n\pi e(-1)^n + n\pi) \right]$$

$$\text{Therefore } f(x) = 2\pi \sum_{n=1}^{\infty} \frac{n(1-e(-1)^n)}{1+n^2\pi^2} \sin n\pi x$$

Example – 4:- Determine the half range Fourier cosine series $f(x) = \sin x$

in $0 < x < \pi$

Ans:- Given that $f(x) = \sin x$ is defined in $(0, \pi)$ i.e. $L = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} \right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx) \quad \text{after putting } L = \pi$$

Now

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \sin x dx = \frac{2}{\pi} [-\cos x]_0^{\pi} = \frac{2}{\pi}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx dx \\ &= -\frac{2}{\pi(n^2-1)} [-(-1)^n - 1] \quad \text{provided } n \neq 1 \end{aligned}$$

By definition $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$

Hence, the half range cosine series is given by

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1} (1 + (-1)^n)$$

i.e. $\sin x = \frac{2}{\pi} - \frac{4}{\pi} \left(\cos \frac{2x}{1 \cdot 3} + \cos \frac{4x}{3 \cdot 5} + \cos \frac{6x}{5 \cdot 7} + \dots \right)$ is the required cosine series.

Exercise 2.3

Are the following functions odd, even, neither odd nor even

(a) $x^2 \cos x$ (b) $\sin x + \cos x$

(c) $f(x) = e^{x^2}$

(d) $\ln x$

(e) $e^{|x|}$

(f) $f(x) = x^4$ ($0 < x < 2\pi$)

$$(g) f(x) = \begin{cases} \cos^2 x, & \text{if } -\pi < x < 0 \\ \sin^2 x, & \text{if } 0 < x < \pi \end{cases} \quad (h) e^{-|x|} \quad (-\pi < x < \pi)$$

2. Determine half range Fourier sine series for $f(x)=x(\pi-x)$ in $0 < x < \pi$

hence deduce $\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \dots = \pi^3/32$.

3. Determine half range Fourier sine series for $f(x)=e^x$ in $0 < x < \pi$

4. Determine the half range Fourier sine series for

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$$

5. Determine the half range Fourier cosine series for

$$f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$$

6. Find half-range Fourier cosine series for $f(x)=x$ in $(0 < x < 2\pi)$