

# 26

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## The Inverse Laplace Transform

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We now know how to find Laplace transforms of “unknown” functions satisfying various initial-value problems. Of course, it’s not the transforms of those unknown function which are usually of interest. It’s the functions, themselves, that are of interest. So let us turn to the general issue of finding a function  $y(t)$  when all we know is its Laplace transform  $Y(s)$ .

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### 26.1 Basic Notions On Recovering a Function from Its Transform

In attempting to solve the differential equation in example 25.1, we got

$$Y(s) = \frac{4}{s-3} ,$$

which, since

$$Y(s) = \mathcal{L}[y(t)]|_s \quad \text{and} \quad \frac{4}{s-3} = \mathcal{L}[4e^{3t}]|_s ,$$

we rewrote as

$$\mathcal{L}[y(t)] = \mathcal{L}[4e^{3t}] .$$

From this, seemed reasonable to conclude that

$$y(t) = 4e^{3t} .$$

But, what if there were another function  $f(t)$  with the same transform as  $4e^{3t}$ ? Then we could not be sure whether the above  $y(t)$  should be  $4e^{3t}$  or that other function  $f(t)$ . Fortunately, someone has managed to prove the following:

**Theorem 26.1 (uniqueness of the transforms)**

Suppose  $f$  and  $g$  are any two piecewise continuous functions on  $[0, \infty)$  of exponential order and having the same Laplace transforms,

$$\mathcal{L}[f] = \mathcal{L}[g] .$$

Then, as piecewise continuous functions,

$$f(t) = g(t) \quad \text{on} \quad [0, \infty) .$$

(You may want to quickly review the discussion of “equality of piecewise continuous functions” on page 497.)

This theorem actually follows from another result that will be briefly discussed at the end of this section. What is important, now, is that this theorem assures us that, if

$$\mathcal{L}[y(t)]|_s = \mathcal{L}[4e^{3t}]|_s ,$$

then

$$y(t) = 4e^{3t} ,$$

at least for  $t \geq 0$ .

What about for  $t < 0$ ? Well, keep in mind that the Laplace transform of any function  $f$ ,

$$F(s) = \mathcal{L}[f]|_s = \int_0^{\infty} f(t)e^{-st} dt ,$$

involves integration only over the positive  $T$ -axis. The behavior of  $f$  on the negative  $T$ -axis has no effect on the formula for  $F(s)$ . In fact,  $f(t)$  need not even be defined for  $t < 0$ . So, even if they exist, there can be no way to recover the values of  $f(t)$  on the negative  $T$ -axis from  $F(s)$ . But that is not a real concern because we will just use the Laplace transform for problems over the positive  $T$ -axis — problems in which we have initial values at  $t = 0$  and want to know what happens *later*.

What all this means is that we are only interested in functions of  $t$  with  $t \geq 0$ . That was hinted at when we began our discussions of the Laplace transform (see note 3 on page 477), but we did not make an issue of it to avoid getting too distracted by technical details. Now, with the inverse transform, requiring  $t \geq 0$  becomes more of an issue. Still, there is no need to obsess about this any more than necessary, or to suddenly start including “for  $t \geq 0$ ” with every formula of  $t$ . Let us just agree that the negative  $T$ -axis is irrelevant to our discussions, and that

*in all formulas involving  $t$ , it is assumed that  $t \geq 0$ .*

► **Example 26.1:** Somewhere above, we have

$$y(t) = 4e^{3t} .$$

What we really mean is that

$$y(t) = 4e^{3t} \quad \text{for } t \geq 0 .$$

We have no idea what  $y(t)$  is for  $t < 0$ . We don't even know whether, in whatever application this may have arisen, it makes sense to talk about  $y(t)$  for  $t < 0$ , nor do we care.<sup>1</sup>

## The Inverse Laplace Transform Defined

We can now officially define the inverse Laplace transform:

Given a function  $F(s)$ , the *inverse Laplace transform of  $F$* , denoted by  $\mathcal{L}^{-1}[F]$ , is that function  $f$  whose Laplace transform is  $F$ .

<sup>1</sup> For example: What if  $y(t)$  denoted the temperature in a cup of coffee  $t$  minutes after being poured? Does it make sense to consider the temperature of the coffee before it exists? (Answer this assuming you are not a Zen master.)

More succinctly:

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t \iff \mathcal{L}[f(t)]|_s = F(s) .$$

Our theorem on uniqueness (theorem 26.1) (along with our understanding about “always assuming  $t \geq 0$ ”) assures us that the above definition for  $\mathcal{L}^{-1}[F]$  is unambiguous. In this definition, of course, we assume  $F(s)$  can be given as  $\mathcal{L}[f(t)]$  for some function  $f$ .

!► **Example 26.2:** We have

$$\mathcal{L}^{-1}\left[\frac{4}{s-3}\right]|_t = 4e^{3t}$$

because

$$\frac{4}{s-3} = \mathcal{L}[4e^{3t}]|_s .$$

Likewise, since

$$\mathcal{L}[t^3]|_s = \frac{6}{s^4} ,$$

we have

$$t^3 = \mathcal{L}^{-1}\left[\frac{6}{s^4}\right]|_t .$$

The fact that

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t \iff \mathcal{L}[f(t)]|_s = F(s)$$

means that any table of Laplace transforms (such as table 24.1 on page 484) is also a table of inverse Laplace transforms. Instead of reading off the  $F(s)$  for each  $f(t)$  found, read off the  $f(t)$  for each  $F(s)$ .

As you may have already noticed, we take inverse transforms of “functions of  $s$  that are denoted by upper case Roman letters” and obtain “functions of  $t$  that are denoted by the corresponding lower case Roman letter”. These notational conventions are consistent with the notational conventions laid down earlier for the Laplace transform.

We should also note that the phrase “inverse Laplace transform” can refer to either the ‘inverse transformed function’  $f$  or to the process of computing  $f$  from  $F$ .

By the way, there is a formula for computing inverse Laplace transforms. If you must know, it is

$$\mathcal{L}^{-1}[F(s)]|_t = \frac{1}{2\pi} \lim_{Y \rightarrow +\infty} \int_{-Y}^Y e^{t(\sigma+i\xi)} F(\sigma+i\xi) d\xi .$$

The integral here is over a line in the complex plane, and  $\sigma$  is a suitably chosen positive value. In deriving this formula, you actually verify uniqueness theorem 26.1. Unfortunately, deriving and verifying this formula goes beyond our current abilities.<sup>2</sup>

Don’t pretend to understand this formula, and don’t try to use it until you’ve had a course in complex variables. Besides, it is not nearly as useful as a good table of transforms.

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<sup>2</sup> Two derivations can be found in third edition of **Transforms and Applications Handbook** (Ed: A. Poularikas, CRC Press). One, using Fourier transforms, is in section 2.4.6 of the chapter on Fourier transforms by Howell. The other, using results from the theory of complex analytic functions, is in section 5.6 of the chapter on Laplace transforms by Poularikas and Seely.

## 26.2 Linearity and Using Partial Fractions

### Linearity of the Inverse Transform

The fact that the inverse Laplace transform is linear follows immediately from the linearity of the Laplace transform. To see that, let us consider  $\mathcal{L}^{-1}[\alpha F(s) + \beta G(s)]$  where  $\alpha$  and  $\beta$  are any two constants and  $F$  and  $G$  are any two functions for which inverse Laplace transforms exist. Following our conventions, we'll denote those inverse transforms by  $f$  and  $g$ . That is,

$$f(t) = \mathcal{L}^{-1}[F(s)]|_t \quad \text{and} \quad g(t) = \mathcal{L}^{-1}[G(s)]|_t .$$

Remember, this is completely the same as stating that

$$\mathcal{L}[f(t)]|_s = F(s) \quad \text{and} \quad \mathcal{L}[g(t)]|_s = G(s) .$$

Because we already know the Laplace transform is linear, we know

$$\mathcal{L}[\alpha f(t) + \beta g(t)]|_s = \alpha \mathcal{L}[f(t)]|_s + \beta \mathcal{L}[g(t)]|_s = \alpha F(s) + \beta G(s) .$$

This, along the definition of the inverse transform and the above definitions of  $f$  and  $g$ , yields

$$\mathcal{L}^{-1}[\alpha F(s) + \beta G(s)]|_t = \alpha f(t) + \beta g(t) = \alpha \mathcal{L}^{-1}[F(s)]|_t + \beta \mathcal{L}^{-1}[G(s)]|_t .$$

Redoing these little computations with as many functions and constants as desired then gives us the next theorem:

#### **Theorem 26.2 (linearity of the inverse Laplace transform)**

The inverse Laplace transform transform is linear. That is,

$$\begin{aligned} \mathcal{L}^{-1}[c_1 F_1(s) + c_2 F_2(s) + \cdots + c_n F_n(s)] \\ = c_1 \mathcal{L}^{-1}[F_1(s)] + c_2 \mathcal{L}^{-1}[F_2(s)] + \cdots + c_n \mathcal{L}^{-1}[F_n(s)] \end{aligned}$$

when each  $c_k$  is a constant and each  $F_k$  is a function having an inverse Laplace transform.

Let's now use the linearity to compute a few inverse transforms.

**!► Example 26.3:** Let's find

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 9}\right]|_t .$$

We know (or found in table 24.1 on page 484) that

$$\mathcal{L}^{-1}\left[\frac{3}{s^2 + 9}\right]|_t = \sin(3t) ,$$

which is almost what we want. To use this in computing our desired inverse transform, we will combine linearity with one of mathematics' oldest tricks (multiplying by 1 with, in this case,  $1 = \frac{3}{3}$ ):

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 + 9}\right]|_t = \mathcal{L}^{-1}\left[\frac{1}{3} \cdot \frac{3}{s^2 + 9}\right]|_t = \frac{1}{3} \mathcal{L}^{-1}\left[\frac{3}{s^2 + 9}\right]|_t = \frac{1}{3} \sin(3t) .$$

The use of linearity along with ‘multiplying by 1’ will be used again and again. Get used to it.

► **Example 26.4:** Let’s find the inverse Laplace transform of

$$\frac{30}{s^7} + \frac{8}{s-4} .$$

We know

$$\mathcal{L}^{-1}\left[\frac{6!}{s^7}\right]_t = t^6 \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{1}{s-4}\right]_t = e^{4t} .$$

So,

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{30}{s^7} + \frac{8}{s-4}\right]_t &= 30\mathcal{L}^{-1}\left[\frac{1}{s^7}\right]_t + 8\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]_t \\ &= 30\mathcal{L}^{-1}\left[\frac{1}{6!} \cdot \frac{6!}{s^7}\right]_t + 8e^{4t} \\ &= \frac{30}{6!}\mathcal{L}^{-1}\left[\frac{6!}{s^7}\right]_t + 8e^{4t} = \frac{30}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}t^6 + 8e^{4t} , \end{aligned}$$

which, after a little arithmetic, reduces to

$$\mathcal{L}^{-1}\left[\frac{30}{s^7} + \frac{8}{s-4}\right]_t = \frac{1}{24}t^6 + 8e^{4t} .$$

## Partial Fractions

When using the Laplace transform with differential equations, we often get transforms that can be converted via ‘partial fractions’ to forms that are easily inverse transformed using the tables and linearity, as above. This means that the general method(s) of partial fractions are particularly important. By now, you should well-acquainted with using partial fractions — remember, the basic idea is that, if we have a fraction of two polynomials

$$\frac{Q(s)}{P(s)}$$

and  $P(s)$  can be factored into two smaller polynomials

$$P(s) = P_1(s)P_2(s) ,$$

then two other polynomials  $Q_1(s)$  and  $Q_2(s)$  can be found so that

$$\frac{Q(s)}{P(s)} = \frac{Q(s)}{P_1(s)P_2(s)} = \frac{Q_1(s)}{P_1(s)} + \frac{Q_2(s)}{P_2(s)} .$$

Moreover, if (as will usually be the case for us) the degree of  $Q(s)$  is less than the degree of  $P(s)$ , then the degree of each  $Q_k(s)$  will be less than the degree of the corresponding  $P_k(s)$ .

You probably used partial fractions to compute some of the integrals in the earlier chapters of this text. We’ll go through a few examples to both refresh our memories of this technique, and to see how it naturally arises in using the Laplace transform to solve differential equations.

► **Example 26.5:** In exercise 25.1 e on page 523, you found that the Laplace transform of the solution to

$$y'' + 4y = 20e^{4t} \quad \text{with } y(0) = 3 \quad \text{and } y'(0) = 12$$

is

$$Y(s) = \frac{3s^2 - 28}{(s-4)(s^2+4)} .$$

The partial fraction expansion of this is

$$Y(s) = \frac{3s^2 - 28}{(s-4)(s^2+4)} = \frac{A}{s-4} + \frac{Bs+C}{s^2+4}$$

for some constants  $A$ ,  $B$  and  $C$ . There are many ways to find these constants. The basic method is to “undo” the partial fraction expansion by getting a common denominator and adding up the fractions on the right:

$$\begin{aligned} \frac{3s^2 - 28}{(s-4)(s^2+4)} &= \frac{A}{s-4} + \frac{Bs+C}{s^2+4} \\ &= \frac{A(s^2+4)}{(s-4)(s^2+4)} + \frac{(s-4)(Bs+C)}{(s-4)(s^2+4)} \\ &= \dots \\ &= \frac{(A+B)s^2 + (C-4B)s + 4(A-C)}{(s-4)(s^2+4)} . \end{aligned}$$

Cutting out the middle and canceling out the common denominator leads to the equation

$$3 \cdot s^2 + 0 \cdot s - 28 = (A+B)s^2 + (C-4B)s + 4(A-C) ,$$

which, in turn, means that our constants satisfy the three by three system

$$\begin{aligned} 3 &= A + B \\ 0 &= C - 4B \\ -28 &= 4A - 4C . \end{aligned}$$

This is relatively simple system. Solving it however you wish, you obtain

$$A = 1 \quad \text{and} \quad B = 2 \quad \text{and} \quad C = 8 .$$

Hence

$$Y(s) = \frac{A}{s-4} + \frac{Bs+C}{s^2+4} = \frac{1}{s-4} + \frac{2s+8}{s^2+4} ,$$

and

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)]|_t &= \mathcal{L}^{-1}\left[\frac{1}{s-4} + \frac{2s+8}{s^2+4}\right]|_t \\ &= \mathcal{L}^{-1}\left[\frac{1}{s-4}\right]|_t + 2\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]|_t + 8\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]|_t \\ &= e^{4t} + 2\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right]|_t + 8 \cdot \frac{1}{2} \mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right]|_t \\ &= e^{4t} + 2\cos(2t) + 4\sin(2t) . \end{aligned}$$

!► **Example 26.6:** In example 25.2 on page 511 we obtained

$$Y(s) = \frac{16}{(s-2)(s^2-7s+12)} + \frac{6s-38}{s^2-7s+12}$$

and, equivalently,

$$Y(s) = \frac{6s^2 - 50s + 92}{(s-2)(s^2-7s+12)}$$

as the Laplace transform of the solution to some initial-value problem. While we could find partial fraction expansions for each term of the first expression above, it will certainly be more convenient to simply find the single partial fraction expansion for the second expression for  $Y(s)$ . But before attempting that, we should note that one factor in the denominator can be further factored,

$$s^2 - 7s + 12 = (s-3)(s-4) \quad ,$$

giving us

$$Y(s) = \frac{6s^2 - 50s + 92}{(s-2)(s-3)(s-4)} \quad .$$

Now we can seek the partial fraction expansion of  $Y(s)$  :

$$\begin{aligned} \frac{6s^2 - 50s + 92}{(s-2)(s-3)(s-4)} &= \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-4} \\ &= \dots \\ &= \frac{A(s-3)(s-4) + B(s-2)(s-4) + C(s-2)(s-3)}{(s-2)(s-3)(s-4)} \quad . \end{aligned}$$

Cutting out the middle and canceling out the common denominator leaves

$$\begin{aligned} 6s^2 - 50s + 92 \\ = A(s-3)(s-4) + B(s-2)(s-4) + C(s-2)(s-3) \quad . \end{aligned} \quad (26.1)$$

Rather than multiplying out the right side of this equation and setting up the system that  $A$ ,  $B$  and  $C$  must satisfy for this equation to hold (as we did in the previous example), let's find these constants after making clever choices for the value of  $s$  in this last equation.

Letting  $s = 2$  in equation (26.1):

$$\begin{aligned} 6(2^2) - 50 \cdot 2 + 92 \\ = A(2-3)(2-4) + B(2-2)(2-4) + C(2-2)(2-3) \end{aligned}$$

$$\hookrightarrow \quad 16 = 2A + 0B + 0C \quad \implies \quad A = 8 \quad .$$

Letting  $s = 3$  in equation (26.1):

$$\begin{aligned} 6(3^2) - 50 \cdot 3 + 92 \\ = A(3-3)(3-4) + B(3-2)(3-4) + C(3-2)(3-3) \end{aligned}$$

$$\hookrightarrow \quad -4 = 0A - B + 0C \quad \implies \quad B = 4 \quad .$$

Letting  $s = 4$  in equation (26.1):

$$\begin{aligned} 6(4^2) - 50 \cdot 4 + 92 \\ = A(4-3)(4-4) + B(4-2)(4-4) + C(4-2)(4-3) \end{aligned}$$

$$\hookrightarrow -12 = 0A + 0B + 2C \implies C = -6 .$$

Combining the above results, we have

$$\begin{aligned} Y(s) &= \frac{6s^2 - 50s + 76}{(s-2)(s-3)(s-4)} \\ &= \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-4} = \frac{8}{s-2} + \frac{4}{s-3} - \frac{6}{s-4} . \end{aligned}$$

Hence,

$$\begin{aligned} y(t) = \mathcal{L}^{-1}[Y(s)]|_t &= \mathcal{L}^{-1}\left[\frac{8}{s-2} + \frac{4}{s-3} - \frac{6}{s-4}\right]|_t \\ &= 8\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]|_t + 4\mathcal{L}^{-1}\left[\frac{1}{s-3}\right]|_t - 6\mathcal{L}^{-1}\left[\frac{1}{s-4}\right]|_t \\ &= 8e^{2t} + 4e^{3t} - 6e^{4t} . \end{aligned}$$

Do recall how to deal with repeated factors in the denominator. In particular, if your denominator has factors of the form

$$(s+c)^n \quad \text{or} \quad (s^2+bs+c)^n$$

for some positive integer  $n$  and constants  $b$  and  $c$ , then the corresponding partial fraction expansions are

$$\frac{A_1}{(s+c)^n} + \frac{A_2}{(s+c)^{n-1}} + \frac{A_3}{(s+c)^{n-2}} + \cdots + \frac{A_n}{s+c}$$

and

$$\frac{A_1s+B_1}{(s^2+bs+c)^n} + \frac{A_2s+B_2}{(s^2+bs+c)^{n-1}} + \frac{A_3s+B_3}{(s^2+bs+c)^{n-2}} + \cdots + \frac{A_ns+B_n}{s^2+bs+c} ,$$

respectively.

!► **Example 26.7:** The partial fraction expansion of

$$Y(s) = \frac{2s^2}{(s-6)^3}$$

is of the form

$$\frac{A}{(s-6)^3} + \frac{B}{(s-6)^2} + \frac{C}{s-6} .$$

To find the constants  $A$ ,  $B$  and  $C$ , we proceed as in the previous examples:

$$\begin{aligned} \frac{2s^2}{(s-6)^3} &= \frac{A}{(s-6)^3} + \frac{B}{(s-6)^2} + \frac{C}{s-6} \\ &= \frac{A}{(s-6)^3} + \frac{B(s-6)}{(s-6)^2(s-6)} + \frac{C(s-6)^2}{(s-6)(s-6)^2} \\ &= \frac{A + B(s-6) + C(s-6)^2}{(s-6)^3} . \end{aligned}$$



So we must have

$$2s^2 = A + B(s - 6) + C(s - 6)^2 .$$

The value of  $A$  can be easily found by letting  $s = 6$  in this equation, and the values of  $B$  and  $C$  can be found by letting  $s = 6$  after taking derivatives of both sides of this equation. Or we can multiply out the right side and rewrite the left side more explicitly, obtaining

$$2s^2 + 0s + 0 = Cs^2 + (B - 12C)s + (A - 6B + 36C) .$$

This tells us that the constants can be obtained by solving the system

$$\begin{aligned} C &= 2 \\ B - 12C &= 0 \\ A - 6B + 36C &= 0 . \end{aligned}$$

In either case, you will discover that

$$A = 72 , \quad B = 24 \quad \text{and} \quad C = 2 .$$

Thus,

$$\begin{aligned} Y(s) &= \frac{2s^2}{(s - 6)^3} \\ &= \frac{A}{(s - 6)^3} + \frac{B}{(s - 6)^2} + \frac{C}{s - 6} \\ &= \frac{72}{(s - 6)^3} + \frac{24}{(s - 6)^2} + \frac{2}{s - 6} . \end{aligned}$$

In the next section, we will discuss an easy way to find the inverse transform of each of the terms in this partial fraction expansion.

## 26.3 Inverse Transforms of Shifted Functions

All the identities derived for the Laplace transform can be rewritten in terms of the inverse Laplace transform. Of particular value to us is the first shifting identity

$$\mathcal{L}[e^{at} f(t)]|_s = F(s - a)$$

where  $F = \mathcal{L}[f(t)]$  and  $a$  is any fixed real number. In terms of the inverse transform, this is

$$\mathcal{L}^{-1}[F(s - a)]|_t = e^{at} f(t) .$$

where  $f = \mathcal{L}^{-1}[F(s)]$  and  $a$  is any fixed real number. Viewed this way, we have a nice way to find inverse transforms of functions that can be written as “shifts” of functions in our tables.

!► **Example 26.8:** Consider

$$\mathcal{L}^{-1}\left[\frac{1}{(s - 6)^3}\right]|_t .$$

Here, the ‘shift’ is clearly by  $a = 6$ , and we have, by the above identity,

$$\mathcal{L}^{-1}\left[\frac{1}{(s-6)^3}\right]\Big|_t = \mathcal{F}^{-1}[F(s-6)]\Big|_t = e^{6t} f(t) \quad . \quad (26.2)$$

We now need to figure out the  $f(t)$  from the fact that

$$F(s-6) = \frac{1}{(s-6)^3} \quad .$$

Letting  $X = s - 6$  in this equation, we have

$$F(X) = \frac{1}{X^3} \quad .$$

Thus,

$$F(s) = \frac{1}{s^3} \quad ,$$

and

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)]\Big|_t = \mathcal{L}^{-1}\left[\frac{1}{s^3}\right]\Big|_t \\ &= \mathcal{L}^{-1}\left[\frac{1}{2!} \cdot \frac{2!}{s^{2+1}}\right]\Big|_t = \frac{1}{2!} \mathcal{L}^{-1}\left[\frac{2!}{s^{2+1}}\right]\Big|_t = \frac{1}{2} t^2 \quad . \end{aligned}$$

Plugging this back into equation (26.2), we obtain

$$\mathcal{L}^{-1}\left[\frac{1}{(s-6)^3}\right]\Big|_t = \dots = e^{6t} f(t) = e^{6t} \frac{1}{2} t^2 = \frac{1}{2} t^2 e^{6t} \quad .$$

In many cases, determining the shift is part of the problem.

**!► Example 26.9:** Consider finding the inverse Laplace transform of

$$\frac{1}{s^2 - 8s + 25} \quad .$$

If the denominator could be factored nicely, we would use partial fractions. This denominator does not factor nicely (unless we use complex numbers). When that happens, try “completing the square” to rewrite the denominator in terms of “ $s - a$ ” for some constant  $a$ . Here,

$$\begin{aligned} s^2 - 8s + 25 &= s^2 - 2 \cdot 4s + [4^2 - 4^2] + 25 \\ &= \underbrace{s^2 - 2 \cdot 4s + 4^2}_{(s-4)^2} - 4^2 + 25 = (s-4)^2 + 9 \quad . \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{s^2 - 8s + 25}\right]\Big|_t &= \mathcal{L}^{-1}\left[\frac{1}{(s-4)^2 + 9}\right]\Big|_t \\ &= \mathcal{L}^{-1}[F(s-4)]\Big|_t = e^{4t} f(t) \quad . \end{aligned} \quad (26.3)$$

Again, we need to find  $f(t)$  from a shifted version of its transform. Here,

$$F(s-4) = \frac{1}{(s-4)^2 + 9} \quad .$$

Letting  $X = s - 4$  in this equation, we have

$$F(X) = \frac{1}{X^2 + 9} ,$$

which means the formula for  $F(s)$  is

$$F(s) = \frac{1}{s^2 + 9} .$$

Thus,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)]|_t = \mathcal{L}^{-1}\left[\frac{1}{s^2 + 9}\right]|_t \\ &= \mathcal{L}^{-1}\left[\frac{1}{3} \cdot \frac{3}{s^2 + 9}\right]|_t = \frac{1}{3} \mathcal{L}^{-1}\left[\frac{3}{s^2 + 3^2}\right]|_t = \frac{1}{3} \sin(3t) . \end{aligned}$$

Plugging this back into equation (26.3), we get

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 - 8s + 25}\right]|_t = \dots = e^{4t} f(t) = e^{4t} \frac{1}{3} \sin(3t) = \frac{1}{3} e^{4t} \sin(3t) .$$

## Additional Exercises

**26.1.** Using the tables (mainly, table 24.1 on page 484) or your own memory, find the inverse Laplace transform for each of the following:

**a.**  $\frac{1}{s - 6}$

**b.**  $\frac{1}{s + 2}$

**c.**  $\frac{1}{s^2}$

**d.**  $\frac{6}{s^4}$

**e.**  $\frac{5}{s^2 + 25}$

**f.**  $\frac{s}{s^2 + 3\pi^2}$

**26.2.** Using the tables and ‘linearity’, find the inverse Laplace transform for each of the following:

**a.**  $\frac{6}{s + 2}$

**b.**  $\frac{1}{s^4}$

**c.**  $\frac{3}{\sqrt{s}} - \frac{8}{s - 4}$

**d.**  $\frac{4s^2 - 4}{s^5}$

**e.**  $\frac{3s + 1}{s^2 + 25}$

**f.**  $\frac{1 - e^{-4s}}{s}$

**26.3.** Verify the following inverse Laplace transforms assuming  $\omega$  is any real constant:

**a.**  $\mathcal{L}^{-1}\left[\frac{s}{(s^2 + \omega^2)^2}\right]|_t = \frac{t}{2\omega} \sin(\omega t)$

**b.**  $\mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)^2}\right]|_t = \frac{1}{2\omega^3} [\sin(\omega t) - \omega t \cos(\omega t)]$

**26.4.** Solve each of the following initial-value problems using the Laplace transform:

**a.**  $y' + 9y = 0$  with  $y(0) = 4$

**b.**  $y'' + 9y = 0$  with  $y(0) = 4$  and  $y'(0) = 6$

**26.5.** Using the tables and partial fractions, find the inverse Laplace transform for each of the following:

a.  $\frac{7s + 5}{(s + 2)(s - 1)}$

b.  $\frac{s - 1}{s^2 - 7s + 12}$

c.  $\frac{1}{s^2 - 4}$

d.  $\frac{3s^2 + 6s + 27}{s^3 + 9s}$

e.  $\frac{1}{s^3 - 4s^2}$

f.  $\frac{8s^3}{s^4 - 81}$

g.  $\frac{5s^2 + 6s - 40}{(s + 6)(s^2 + 16)}$

h.  $\frac{2s^3 + 3s^2 + 2s + 27}{(s^2 + 9)(s^2 + 1)}$

i.  $\frac{6s^2 + 62s + 92}{(s + 1)(s^2 + 10s + 21)}$

**26.6.** Solve each of the following initial-value problems using the Laplace transform (and partial fractions):

a.  $y'' - 9y = 0$  with  $y(0) = 4$  and  $y'(0) = 9$

b.  $y'' + 9y = 27t^3$  with  $y(0) = 0$  and  $y'(0) = 0$

c.  $y'' + 8y' + 7y = 165e^{4t}$  with  $y(0) = 8$  and  $y'(0) = 1$

**26.7.** Using the translation identity (and the tables), find the inverse Laplace transform for each of the following:

a.  $\frac{1}{(s - 7)^5}$

b.  $\frac{1}{s^2 - 6s + 45}$

c.  $\frac{s}{s^2 - 6s + 45}$

d.  $\frac{1}{\sqrt{s + 2}}$

e.  $\frac{1}{s^2 + 8s + 16}$

f.  $\frac{s}{s^2 - 12s + 40}$

g.  $\frac{1}{s^2 + 12s + 40}$

h.  $\frac{s^2}{(s - 3)^5}$

**26.8.** Using the Laplace transform with the translation identity, solve the following initial-value problems:

a.  $y'' - 8y' + 17y = 0$  with  $y(0) = 3$  and  $y'(0) = 12$

b.  $y'' - 6y' + 9y = e^{3t}t^2$  with  $y(0) = 0$  and  $y'(0) = 0$

c.  $y'' + 6y' + 13y = 0$  with  $y(0) = 2$  and  $y'(0) = 8$

d.  $y'' + 8y' + 17y = 0$  with  $y(0) = 3$  and  $y'(0) = -12$

**26.9.** Using the Laplace transform, solve the following initial-value problems:

a.  $y'' = e^t \sin(t)$  with  $y(0) = 0$  and  $y'(0) = 0$

b.  $y'' - 4y' + 40y = 122e^{-3t}$  with  $y(0) = 0$  and  $y'(0) = 8$

c.  $y'' - 9y = 24e^{-3t}$  with  $y(0) = 6$  and  $y'(0) = 2$

d.  $y'' - 4y' + 13y = e^{2t} \sin(3t)$  with  $y(0) = 4$  and  $y'(0) = 3$

**26.10.** The inverse transforms of the following could be computed using partial fractions. Instead, find the inverse transform of each using the appropriate integration identity from section 25.3.

a.  $\frac{1}{s(s^2 + 9)}$

b.  $\frac{1}{s(s - 4)}$

c.  $\frac{1}{s(s - 3)^2}$