Lec-5

TRANSFORMS OF DERIVATIVES:

Suppose that $y(t)$ is continuous for all $t \geq 0$ satisfies the condition $|y(t)| \leq Me^{-kt}$ for some k and M, and has a derivative $y'(t)$that is piecewise continuous on every finite interval in the range $t \geq 0$. then the Laplace transform of the derivative $y'(t)$ exists when $\boldsymbol{s} > k$ and $L(y') = sL\{y(t)\} - y(0)$.

Similarly $L(y'') = sL\{y'(t)\} - y(0) = s[sL\{y'(t)\} - y(0)] - y'(0)$

$=s^2 L(y) - sy(0) - y'(0)$.

Therefore Derivative of nth order is given by

$$L\left(y^{(n)}\right) = s^n L\{y(t)\} - s^{n-1}y(0) - s^{n-2}y(0) - \cdots \ldots \ldots \ldots \ldots .. - y^{(n-1)}(0)$$

APPLICATION OF TRANSFORM OF DERIVATIVES:

1. It is applicable for solving linear differential equation with constant coefficients.

2. it is applicable for solving first/higher order initial value problems as well as solution of boundary value problems.

SOLUTION OF DIFFERENTIAL EQUATIONS, INITIAL VALUE PROBLEMS:

Step1: Take the Laplace transform of both sides of the given differential equation and put the formula of $L(y'), L(y''),$……….. $L\left(y^{(n)}\right)$and the given initial conditions.

Step2: Transpose the terms with minus signs to the right. Divide by the coefficient of $L\{y(t)\}$,getting $L\{f(t)\}$ as a known function of s.

Step3: Resolve this function of s in partial fractions and take the inverse transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

WORKED OUT EXAMPLES:

Example3: Solve the following differential equation by the method of transforms.

$$y'' + y' - 6y = 1\,, \quad y(0) = 1, y'(0) = 1$$
$$y'' + y = 3cos2t\,, \quad y(0) = 2, y'(0) = 0$$

$$y'' + 2y' - 3y = \sin t\ , \quad y(0) = 0, \qquad y'(0) = 0$$
$$y'' + 6y' + 8y = e^{-3t} - e^{-5t}\ , \qquad y(0) = 0, y'(0) = 0$$

Solution: (a) Given that $y'' + y' - 6y = 1$ ....(1) , $y(0) = 1,\qquad y'(0) = 1$ Taking Laplace transform both sides of equation (1)and apply the formula for the derivatives, we get $L(\, y'' + y' - 6y) = L(1)$

$$L(y'') + L(y') - 6L(y) = L(1)$$
$$\Rightarrow s^2L\{y(t)\} - sy(0) - y'(0) + sL\{y(t)\} - y(0) - 6L\{y(t)\} = \frac{1}{s}$$
$$\Rightarrow (s^2 + s - 6)L\{y(t)\} = \frac{1}{s} + s + 2 = \frac{s^2 + 2s + 1}{s}$$
$$\Rightarrow L\{y(t)\} = \frac{s^2 + 2s + 1}{s(s-2)(s+3)} \Rightarrow y(t) = L^{-1}\left(\frac{s^2 + 2s + 1}{s(s-2)(s+3)}\right)$$

Using partial fraction, we get

$$\frac{s^2 + 2s + 1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

$s^2 + 2s + 1 = A(s-2)(s+3) + Bs(s+3) + cs(s-2)$.....................................(2)

$Puts = 0, s = 2, s = -3\ inequation$ (2)Respectively we get the value of A,B,C.

$$i.eA = \frac{-1}{6}\ \ , B = \frac{4}{15}\ , C = \frac{9}{10}$$

Therefore A,B,C of a corresponding $y(t)$ is given by

$$y(t) = L^{-1}\left(\frac{s^2 + 2s + 1}{s(s-2)(s+3)}\right) = L^{-1}\left(\frac{\frac{-1}{6}}{s} + \frac{\frac{4}{15}}{s-2} + \frac{\frac{9}{10}}{s+3}\right)$$
$$y(t) = \frac{-1}{6} + \frac{4}{15}e^{2t} + \frac{9}{10}e^{-3t}$$

(b) **)** Given that $y'' + y = 3cos2t$ ....(1) , $y(0) = 2,\qquad y'(0) = 0$

Taking Laplace transform both sides of equation (1)and apply the formula for the derivatives ,we get $L(\, y'' + y) = L(3cos2t)$

$$\Rightarrow L(y'') + L(y) = L(3cos2t)$$
$$\Rightarrow s^2L\{y(t)\} - sy(0) - y'(0) + L\{y(t)\} = \frac{3s}{s^2 + 4}$$

$$\Rightarrow (s^2+1)L\{y(t)\} = \frac{3s}{s^2+4} + 2s \Rightarrow L\{y(t)\} = \frac{3s}{(s^2+4)(s^2+1)} + \frac{2s}{s^2+1}$$

$$L\{y(t)\} = \frac{s}{s^2+1} - \frac{s}{s^2+4} + \frac{2s}{s^2+1} \quad \Rightarrow y(t) = L^{-1}\left(\frac{3s}{s^2+1} - \frac{s}{s^2+4}\right)$$

$$y(t) = 3cost - cos2t.$$

(c) **)** Given that $y'' + 2y' - 3y = sint$ ……………(1) , $y(0) = 0, y'(0) = 0$

Taking Laplace transform both sides of equation (1)and apply the formula for the derivatives ,we get $L(\, y'' + 2y' - 3y) = L(sint)$

$$\Rightarrow L(y'') + 2L(y') - 3L(y) = L(sint)$$

$$\Rightarrow s^2L\{y(t)\} - sy(0) - y'(0) + 2[sL\{y(t)\} - y(0)] - 3L\{y(t)\} = \frac{1}{s^2+1}$$

$$\Rightarrow (s^2+2s-3)L\{y(t)\} = \frac{1}{s^2+1} \Rightarrow L\{y(t)\} = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$y(t) = L^{-1}\left(\frac{1}{(s^2+1)(s^2+2s-3)}\right)$$

Using partial fraction, we get

$$\frac{1}{(s^2+1)(s^2+2s-3)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s-3}$$

i.e $1 = (As+B)(s^2+2s-3) + (Cs+D)(s^2+1)$

$$1 = (A+C)s^3 + (2A+B+D)s^2 + (2B+C)s + (D-3B)$$

Equating the coefficients of $s^3, s^2, sandconstantterms$ both sides, we get

$$A = \frac{1}{4}\ , \quad B = \frac{1}{8}\ , C = \frac{-1}{4}\ , D = \frac{-5}{8}$$

Therefore $y(t) = L^{-1}\left(\frac{1}{(s^2+1)(s^2+2s-3)}\right) = L^{-1}\left(\frac{\frac{1}{4}s+\frac{1}{8}}{s^2+1}\right) - L^{-1}\left(\frac{\frac{1}{4}s+\frac{5}{8}}{s^2+2s-3}\right)$

$y(t) = \frac{1}{4}cost + \frac{1}{8}sint - \frac{1}{16}e^{-t}(4cosh2t - 3sinh2t).$

(d) Given that $y'' + 6y' + 8y = e^{-3t} - e^{-5t}$………………(1) , $y(0) = 0, y'(0) = 0$

Taking Laplace transform both sides of equation (1)and apply the formula for the derivatives ,we get $L(\, y'' + 6y' + 8y) = L(e^{-3t} - e^{-5t})$

$$\Longrightarrow s^2L\{y(t)\} - sy(0) - y'(0) + 6[sL\{y(t)\} - y(0)] + 8L\{y(t)\} = \frac{1}{s+3} - \frac{1}{s+5}$$

$$\Longrightarrow (s^2 + 6s + 8)L\{y(t)\} = \frac{1}{(s+5)(s+3)}$$

$$\Rightarrow L\{y(t)\} = \frac{1}{(s+5)(s+3)(s+2)(s+4)}$$

$$\{y(t)\} = L^{-1}\left(\frac{1}{(s+5)(s+3)(s+2)(s+4)}\right)$$

Using partial fraction, we get

$$\frac{1}{(s+2)(s+3)(s+4)(s+5)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4} + \frac{D}{s+5}$$

$1 = A(s+3)(s+4)(s+5) + B(s+2)(s+4)(s+5) + C(s+2)(s+3)(s+5) + D(s+3)(s+4)(s+2)$ ……………………………………………………………………(2)

$Puts = -2, s = -3, s = -4\ ands = -5\ inequation(2)$respectively we get the value of A,B,C & D.

$$A = \frac{1}{6}\ ,\quad B = \frac{-1}{2}\ ,C = \frac{1}{2}\ ,D = \frac{-1}{6}$$

Therefore $y(t) = L^{-1}\left(\frac{1}{(s+2)(s+3)(s+4)(s+5)}\right) =$

$$\frac{1}{6}L^{-1}\left(\frac{1}{s+2}\right) - \frac{1}{2}L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+4}\right) - \frac{1}{6}L^{-1}\left(\frac{1}{s+5}\right)$$

$$y(t) = \frac{1}{6}e^{-2t} - \frac{1}{2}e^{-3t} + \frac{1}{2}e^{-4t} - \frac{1}{6}e^{-5t}$$