

**SECOND SHIFTING THEOREM:**

If  $L\{f(t)\} = F(s)$  and  $u(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$  then we define a new function

$$\overline{f(t)} = f(t - a)u(t - a) = \begin{cases} 0 & \text{if } t < a \\ f(t - a) & \text{if } t > a \end{cases}$$

$$L\{f(t - a)u(t - a)\} = \int_0^\infty e^{-st} f(t - a)u(t - a) dt$$

$$= \int_0^a e^{-st} f(t - a)u(t - a) dt + \int_a^\infty e^{-st} f(t - a)u(t - a) dt$$

$$= \int_a^\infty e^{-s(u+a)} f(u) du \quad \text{where } t - a = u \text{ and } u(t - a) = 1$$

$$\Rightarrow e^{-sa} \int_a^\infty e^{-su} f(u) dt = e^{-sa} F(s)$$

Therefore  $L\{f(t - a)u(t - a)\} = e^{-sa} F(s)$  and  $L^{-1}\{e^{-sa} F(s)\} = f(t - a)u(t - a)$

**WORKED OUT EXAMPLES:**

**Example5:** Find Laplace transform of the following functions

- (a)  $(t - 1)^2 u(t - 2)$       (b)  $4u(t - \pi) \cos t$  (c)  $9u(t - \pi) \sin t$   
 (d)  $e^{-3t} u(t - 2)$

Solution : (a)  $L\{(t - 1)^2 u(t - 2)\}$

Comparing with second shifting theorem  $u(t - a) = u(t - 2)$  i.e  $a = 2$

$$f(t - a) = f(t - 2) = (t - 1)^2$$

Put  $t = t + 2$  then  $f(t) = (t + 1)^2 = t^2 + 2t + 1$

$$F(s) = L\{t^2 + 2t + 1\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} .$$

Hence by the second shifting theorem, we get  $L\{f(t - a)u(t - a)\} = e^{-sa} F(s)$

$$L\{(t-1)^2 u(t-2)\} = e^{-2s} \left( \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right)$$

$$(b) L\{4u(t-\pi)\cos t\}$$

Comparing with second shifting theorem  $u(t-a) = u(t-\pi)$  i.e  $a = \pi$

$$f(t-a) = f(t-\pi) = 4\cos t$$

$$\text{Put } t = t + \pi \quad \text{then } f(t) = 4\cos(t+\pi) = -4\cos t$$

$$F(s) = L\{-4\cos t\} = \frac{-4s}{s^2+1}.$$

Hence by the second shifting theorem, we get  $L\{f(t-a)u(t-a)\} = e^{-sa} F(s)$

$$L\{4u(t-\pi)\cos t\} = e^{-\pi s} \left( \frac{-4s}{s^2+1} \right).$$

$$(c) L\{9u(t-\pi)\sin t\}$$

Comparing with second shifting theorem  $u(t-a) = u(t-\pi)$  i.e  $a = \pi$

$$f(t-a) = f(t-\pi) = 9\sin t$$

$$\text{Put } t = t + \pi \quad \text{then } f(t) = 9\sin(t+\pi) = -9\sin t$$

$$F(s) = L\{-9\sin t\} = \frac{-9}{s^2+1}.$$

Hence by the second shifting theorem, we get  $L\{f(t-a)u(t-a)\} = e^{-sa} F(s)$

$$L\{9u(t-\pi)\sin t\} = e^{-\pi s} \left( \frac{-9}{s^2+1} \right).$$

$$(d) L\{e^{-3t}u(t-2)\}$$

Comparing with second shifting theorem  $u(t-a) = u(t-2)$  i.e  $a = 2$

$$f(t-a) = f(t-2) = e^{-3t}$$

$$\text{Put } t = t + 2 \quad \text{then } f(t) = e^{-3(t+2)} = e^{-6}e^{-3t}$$

$$F(s) = L\{e^{-6}e^{-3t}\} = \frac{e^{-6}}{s+3}.$$

Hence by the second shifting theorem, we get  $L\{f(t-a)u(t-a)\} = e^{-sa} F(s)$

$$L\{e^{-3t}u(t-2)\} = e^{-2s} \left( \frac{e^{-6}}{s+3} \right)$$