

APPLICATION OF TRANSFORM OF DERIVATIVES:

1. It is applicable for solving linear differential equation with constant coefficients.
2. it is applicable for solving first/higher order initial value problems as well as solution of boundary value problems.

SOLUTION OF DIFFERENTIAL EQUATIONS, INITIAL VALUE PROBLEMS:

Step1: Take the Laplace transform of both sides of the given differential equation and put the formula of $L(y')$, $L(y'')$,..... $L(y^{(n)})$ and the given initial conditions.

Step2: Transpose the terms with minus signs to the right. Divide by the coefficient of $L\{y(t)\}$, getting $L\{f(t)\}$ as a known function of s .

Step3: Resolve this function of s in partial fractions and take the inverse transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

WORKED OUT EXAMPLES:

Example3: Solve the following differential equation by the method of transforms.

$$\begin{aligned}y'' + y' - 6y &= 1, & y(0) &= 1, y'(0) = 1 \\y'' + y &= 3\cos 2t, & y(0) &= 2, y'(0) = 0 \\y'' + 2y' - 3y &= \sin t, & y(0) &= 0, y'(0) = 0 \\y'' + 6y' + 8y &= e^{-3t} - e^{-5t}, & y(0) &= 0, y'(0) = 0\end{aligned}$$

Solution: (a) Given that $y'' + y' - 6y = 1$ (1), $y(0) = 1$, $y'(0) = 1$
Taking Laplace transform both sides of equation (1) and apply the formula for the derivatives, we get $L(y'' + y' - 6y) = L(1)$

$$\begin{aligned}L(y'') + L(y') - 6L(y) &= L(1) \\ \Rightarrow s^2L\{y(t)\} - sy(0) - y'(0) + sL\{y(t)\} - y(0) - 6L\{y(t)\} &= \frac{1}{s} \\ \Rightarrow (s^2 + s - 6)L\{y(t)\} &= \frac{1}{s} + s + 2 = \frac{s^2 + 2s + 1}{s}\end{aligned}$$

$$\Rightarrow L\{y(t)\} = \frac{s^2 + 2s + 1}{s(s-2)(s+3)} \Rightarrow y(t) = L^{-1}\left(\frac{s^2 + 2s + 1}{s(s-2)(s+3)}\right)$$

Using partial fraction, we get

$$\frac{s^2 + 2s + 1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$s^2 + 2s + 1 = A(s-2)(s+3) + Bs(s+3) + cs(s-2) \dots\dots\dots(2)$$

Put $s = 0, s = 2, s = -3$ in equation (2) respectively we get the value of A,B,C.

$$i.e A = \frac{-1}{6}, B = \frac{4}{15}, C = \frac{9}{10}$$

Therefore A,B,C of a corresponding $y(t)$ is given by

$$y(t) = L^{-1}\left(\frac{s^2 + 2s + 1}{s(s-2)(s+3)}\right) = L^{-1}\left(\frac{-1}{6s} + \frac{4}{15(s-2)} + \frac{9}{10(s+3)}\right)$$

$$y(t) = \frac{-1}{6} + \frac{4}{15}e^{2t} + \frac{9}{10}e^{-3t}$$

(b) Given that $y'' + y = 3\cos 2t$ (1), $y(0) = 2, y'(0) = 0$

Taking Laplace transform both sides of equation (1) and apply the formula for the derivatives, we get $L(y'' + y) = L(3\cos 2t)$

$$\Rightarrow L(y'') + L(y) = L(3\cos 2t)$$

$$\Rightarrow s^2L\{y(t)\} - sy(0) - y'(0) + L\{y(t)\} = \frac{3s}{s^2 + 4}$$

$$\Rightarrow (s^2 + 1)L\{y(t)\} = \frac{3s}{s^2 + 4} + 2s \Rightarrow L\{y(t)\} = \frac{3s}{(s^2 + 4)(s^2 + 1)} + \frac{2s}{s^2 + 1}$$

$$L\{y(t)\} = \frac{s}{s^2 + 1} - \frac{s}{s^2 + 4} + \frac{2s}{s^2 + 1} \Rightarrow y(t) = L^{-1}\left(\frac{3s}{s^2 + 1} - \frac{s}{s^2 + 4}\right)$$

$$y(t) = 3\cos t - \cos 2t.$$

(c) Given that $y'' + 2y' - 3y = \sin t$ (1), $y(0) = 0, y'(0) = 0$

Taking Laplace transform both sides of equation (1) and apply the formula for the derivatives, we get $L(y'' + 2y' - 3y) = L(\sin t)$

$$\begin{aligned} &\Rightarrow L(y'') + 2L(y') - 3L(y) = L(\sin t) \\ \Rightarrow s^2 L\{y(t)\} - sy(0) - y'(0) + 2[sL\{y(t)\} - y(0)] - 3L\{y(t)\} &= \frac{1}{s^2 + 1} \\ \Rightarrow (s^2 + 2s - 3)L\{y(t)\} = \frac{1}{s^2 + 1} &\Rightarrow L\{y(t)\} = \frac{1}{(s^2 + 1)(s^2 + 2s - 3)} \\ y(t) = L^{-1}\left(\frac{1}{(s^2 + 1)(s^2 + 2s - 3)}\right) \end{aligned}$$

Using partial fraction, we get

$$\frac{1}{(s^2 + 1)(s^2 + 2s - 3)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s - 3}$$

i.e $1 = (As + B)(s^2 + 2s - 3) + (Cs + D)(s^2 + 1)$

$$1 = (A + C)s^3 + (2A + B + D)s^2 + (2B + C)s + (D - 3B)$$

Equating the coefficients of s^3, s^2, s and constant terms both sides, we get

$$A = \frac{1}{4}, \quad B = \frac{1}{8}, \quad C = \frac{-1}{4}, \quad D = \frac{-5}{8}$$

Therefore $y(t) = L^{-1}\left(\frac{1}{(s^2+1)(s^2+2s-3)}\right) = L^{-1}\left(\frac{\frac{1}{4}s + \frac{1}{8}}{s^2+1}\right) - L^{-1}\left(\frac{\frac{1}{4}s + \frac{5}{8}}{s^2+2s-3}\right)$

$$y(t) = \frac{1}{4} \cos t + \frac{1}{8} \sin t - \frac{1}{16} e^{-t} (4 \cosh 2t - 3 \sinh 2t).$$

(d) Given that $y'' + 6y' + 8y = e^{-3t} - e^{-5t} \dots\dots\dots(1)$, $y(0) = 0, y'(0) = 0$

Taking Laplace transform both sides of equation (1) and apply the formula for the derivatives, we get $L(y'' + 6y' + 8y) = L(e^{-3t} - e^{-5t})$

$$\begin{aligned} \Rightarrow s^2 L\{y(t)\} - sy(0) - y'(0) + 6[sL\{y(t)\} - y(0)] + 8L\{y(t)\} \\ = \frac{1}{s + 3} - \frac{1}{s + 5} \end{aligned}$$

$$\Rightarrow (s^2 + 6s + 8)L\{y(t)\} = \frac{1}{(s + 5)(s + 3)}$$

$$\Rightarrow L\{y(t)\} = \frac{1}{(s + 5)(s + 3)(s + 2)(s + 4)}$$

$$\{y(t)\} = L^{-1} \left(\frac{1}{(s+5)(s+3)(s+2)(s+4)} \right)$$

Using partial fraction, we get

$$\frac{1}{(s+2)(s+3)(s+4)(s+5)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4} + \frac{D}{s+5}$$

$$1 = A(s+3)(s+4)(s+5) + B(s+2)(s+4)(s+5) + C(s+2)(s+3)(s+5) + D(s+3)(s+4)(s+2)$$

.....(2)

Put $s = -2, s = -3, s = -4$ and $s = -5$ in equation (2) respectively we get the value of A, B, C & D.

$$A = \frac{1}{6}, \quad B = \frac{-1}{2}, \quad C = \frac{1}{2}, \quad D = \frac{-1}{6}$$

Therefore $y(t) = L^{-1} \left(\frac{1}{(s+2)(s+3)(s+4)(s+5)} \right) =$

$$\frac{1}{6}L^{-1} \left(\frac{1}{s+2} \right) - \frac{1}{2}L^{-1} \left(\frac{1}{s+3} \right) + \frac{1}{2}L^{-1} \left(\frac{1}{s+4} \right) - \frac{1}{6}L^{-1} \left(\frac{1}{s+5} \right)$$

$$y(t) = \frac{1}{6}e^{-2t} - \frac{1}{2}e^{-3t} + \frac{1}{2}e^{-4t} - \frac{1}{6}e^{-5t}$$