Floyd-Warshall Algorithm

Ms. Sasmita Kumari Nayak
Computer Science & Engineering

Floyd Warshall Algorithm

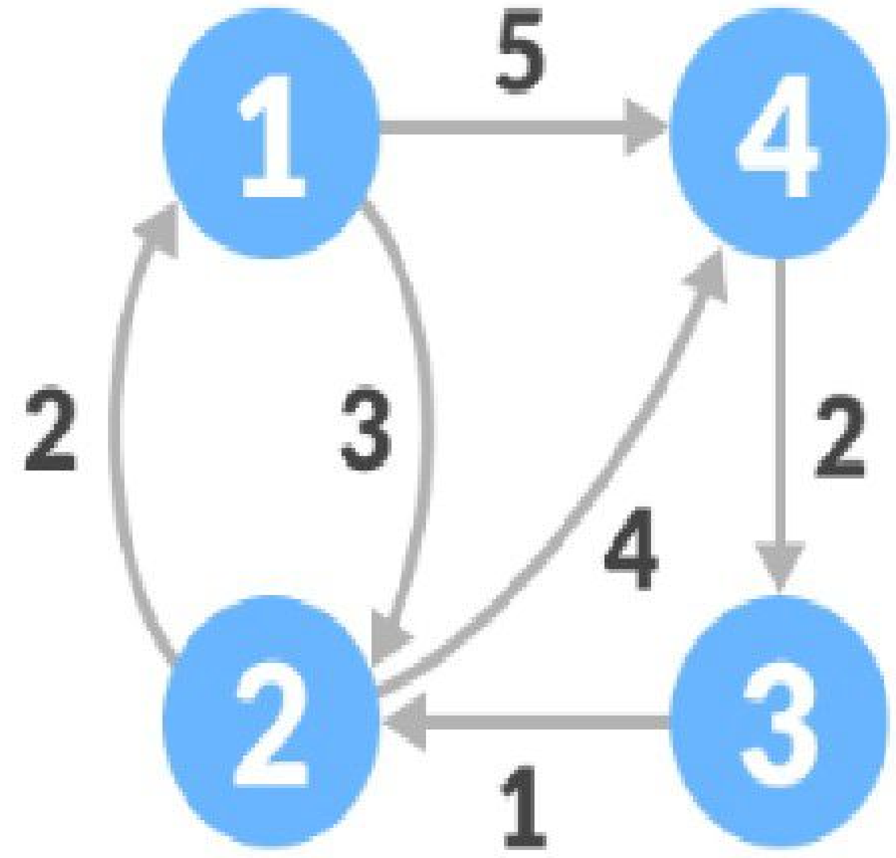
- Floyd Warshall Algorithm is a famous algorithm.
- It is used to solve All Pairs Shortest Path Problem.
- It computes the shortest path between every pair of vertices of the given graph.
- Floyd Warshall Algorithm is an example of dynamic programming approach.

Advantages

- Floyd Warshall Algorithm has the following main advantages-
  - It is extremely simple.
  - It is easy to implement.

PRACTICE PROBLEM

- Consider the following directed weighted graph.
- Using Floyd Warshall Algorithm, find the shortest path distance between every pair of vertices.



Solution

- Follow the steps below to find the shortest path between all the pairs of vertices.

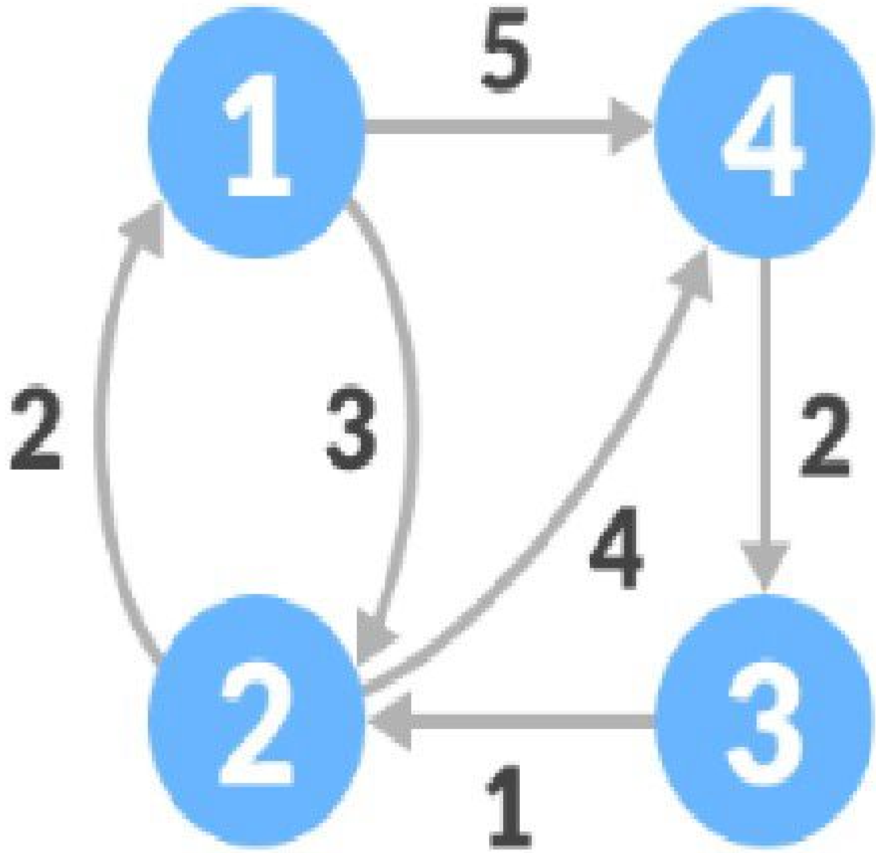
Step-01

- Remove all the self loops and parallel edges (keeping the lowest weight edge) from the graph.
- In the given graph, there are neither self edges nor parallel edges.

Step-02

- Write the initial distance matrix A^0.
- It represents the distance between every pair of vertices in the form of given weights.
- For diagonal elements (representing self-loops), distance value = 0.
- For vertices having a direct edge between them, distance value = weight of that edge.
- For vertices having no direct edge between them, distance value = ∞.

Cont…



$$A^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & \infty & 4 \\ \infty & 1 & 0 & \infty \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

Step-03

- Now, create a matrix A^1 using matrix A^0. The elements in the first column and the first row as well as diagonal elements are left as they are. The remaining cells are filled in the following way.
- Let k be the intermediate vertex in the shortest path from source to destination. In this step, k is the first vertex.

if (A[i][j] > A[i][k] + A[k][j]).

then A[i][j] is filled with (A[i][k] + A[k][j])

Cont…

- That is, if the direct distance from the source to the destination is greater than the path through the vertex k, then the cell is filled with A[i][k] + A[k][j].
- In this step, k is vertex 1. We cacluate the distance from source vertex to destination vertex through this vertex k.

$$A^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & & \\ \infty & & 0 & \\ \infty & & & 0 \end{bmatrix} \end{array} \longrightarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 5 \\ 2 & 0 & 9 & 4 \\ \infty & 1 & 0 & 8 \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

Cont…

- For example:

  For $A^1[2, 4]$, the direct distance from vertex 2 to 4 is 4 and the sum of the distance from vertex 2 to 4 through vertex (i.e. from vertex 2 to 1 and from vertex 1 to 4) is 7.

  $A^1[2, 4] = A^0[2, 1] + A^0[1, 4] = 2 + 5 = 7$

  Since $4 < 7$, $A^0[2, 4]$ is filled with 4.

Step-04

- In a similar way, A^2 is created using A^1. The elements in the second column and the second row are left as they are.
- In this step, k is the second vertex (i.e. vertex 2). The remaining steps are the same as in **step 3**.

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & & \\ 2 & 0 & 9 & 4 \\ & 1 & 0 & \\ & \infty & & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 9 & 5 \\ 2 & 0 & 9 & 4 \\ 3 & 1 & 0 & 5 \\ \infty & \infty & 2 & 0 \end{bmatrix} \end{array}$$

Step-05

- Similarly, A^3 and A^4 is also created.

$$A^3 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & & \infty & \\ & 0 & 9 & \\ \infty & 1 & 0 & 8 \\ & & 2 & 0 \end{bmatrix} \end{array} \longrightarrow \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3 & 9 & 5 \\ 2 & 0 & 9 & 4 \\ 3 & 1 & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

Cont…

- A^4 gives the shortest path between each pair of vertices.

$$A^4 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & & & 5 \\ & 0 & & 4 \\ & & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3 & 7 & 5 \\ 2 & 0 & 6 & 4 \\ 3 & 1 & 0 & 5 \\ 5 & 3 & 2 & 0 \end{bmatrix} \end{array}$$

Algorithm

```
//It represents the distance between every pair of vertices as
given.
//n is the number of vertices present in the graph

Create a n x n matrix

For each cell (i, j) in D do-

    // for all diagonal elements, value = 0

    if i = = j

        D[i][j] = 0

   // If there exists a direct edge between the vertices,
   //value = weight of edge

    if (i, j) is an edge in E

        D[i][j] = weight (i,j)

  // If there is no direct edge between the vertices, value = ∞

    Else

        D[i][j] = infinity
```

```
// Find the distance for all pair shortest path

for k from 1 to n

    for i from 1 to n

        for j from 1 to n

            if D[i][j] > D[i][k] + D[k][j]
                    D[i][j] = D[i][k] + D[k][j]
```

Time complexity

- Floyd Warshall Algorithm consists of three loops over all the nodes.
- The inner most loop consists of only constant complexity operations.
- So, the time complexity of the Floyd-Warshall algorithm is $O(n^3)$.
- Here, n is the number of nodes in the given graph.

Applications of Floyd Warshall Algorithm

- To find the shortest path is a directed graph
- To find the transitive closure of directed graphs
- To find the Inversion of real matrices
- For testing whether an undirected graph is bipartite

Assignment

- Using Floyd Warshall Algorithm, find the shortest path distance between every pair of vertices.