

## CHAPTER-II

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# PROBABILITY THEORY AND PROBABILITY DISTRIBUTIONS

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As a subjective approach, the probability of an event is defined as the degree of one's belief, conviction and experience concerning the likelihood of occurrence of the event in the situation of uncertainty like:

- What is the chance that there will be rain tomorrow?
- What is the likelihood that a given project will be completed in time?
- Probably she will get above 80% marks in the final examination.

Galileo (1564-1642), an Italian mathematician was the first man to attempt at a quantitative measure of probability while dealing with some problems related to the theory of gambling. However, systematic and scientific foundation of mathematical theory of probability was laid in mid seventeenth century by two French mathematicians B. Pascal (1623-62) and Pierry de Fermat (1601-65). To begin the discussion on probability, we need to define the following terms

**Experiment or Trial:** An experiment is any activity that generates data. It is identified by the fact that it has several possible outcomes and all these outcomes are known in advance and can not be predicted with certainty. Such an experiment is called a trial or random experiment. For example, in tossing a fair coin, we cannot be certain whether the outcome will be head or tail.

**Sample Space:** The set of all possible outcomes of an experiment is known as sample space. The possible outcomes are called as sample points or elements of the sample space. It is denoted by  $S$  or  $\Omega$ .

**Event:** A subset of the sample space is called an event. For example, getting an odd number when die is rolled is an event.

**Simple and Compound Event:** A simple or elementary event is a single possible outcome of an experiment. For example, in tossing a coin, the event of coming up head is an elementary event. Compound events have two or more elementary events in it. For example, drawing a black ace from a pack of cards is a compound event since it contains two elementary events of black colour and ace.

**Exhaustive Events:** The total number of possible outcomes of any trial is known as exhaustive events. For example, in throwing a die there are six exhaustive cases since any one of the six faces 1, 2, 3, 4, 5 or 6 may come uppermost while in throwing two dice, the exhaustive cases are  $6^2 = 36$  since any of the 6 numbers 1 to 6 on the first die can associate with any of six numbers on the other die.

**Mutually Exclusive Events:** Events are said to be mutually exclusive if happening of any event precludes the happening of all other, i.e. no two or more events can happen simultaneously in the same trial. For example, in throwing a die all the six faces numbered 1 to 6 are mutually exclusive.

**Equally Likely Events:** Outcomes of a trial are said to be equally likely if there is no reason to expect one in preference to other. For example, in tossing an unbiased/uniform coin head and tail are equally likely events.

**Favourable Events:** The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event. For example, in drawing a card from pack of cards the numbers of cases favourable to drawing an ace are 4 while for drawing a spade are 13 and for drawing a red card are 26.

**Independent Events:** Events are said to be independent if the occurrence or non-occurrence of an event is not affected by the occurrence or non-occurrence of remaining events. For example, if we draw a card from a pack of well shuffled cards and replace it before drawing second, the result of the second draw is independent that of the first draw. But, however, if the first card drawn is not replaced then the result of second draw is dependent on the result of the first draw and the events are called dependent events.

**Complementary Events:** Two events are said to be complementary if they are mutually exclusive and exhaustive for a given sample space. For example, when a die is thrown then events representing an even number and an odd number are complementary events.

**2.1 Definition of Probability:** Here we shall discuss different approaches in calculating the probability of an event.

**Classical Approach (Mathematical Definition):** If a trial results in  $n$  exhaustive, mutually exclusive and equally likely cases and  $m$  of them are favourable to the happening of an event  $E$  then the probability  $p$  of happening of event  $E$  is given by

$$p = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of case}} = \frac{m}{n}$$

Since  $0 \leq m \leq n$ , so  $p$  is non-negative and cannot exceed unity, i.e.  $0 \leq p \leq 1$

If  $p = 1$ , then event  $E$  is called a certain event and if  $p = 0$ ,  $E$  is called an impossible event. However the classical definition of probability is not applicable if i) the various outcomes of the trial are not equally likely, ii) exhaustive number of cases in a trial is infinite.

**Relative Frequency Approach (Statistical/Empirical Definition):** If a trial is repeated a number of times essentially under homogenous and identical conditions, then the limiting value of the ratio of number of times the event happens to the total number of trials, as the trials become indefinitely large, is called the probability of happening of that event. So if in  $n$  trials, an event  $E$  happens  $m$  times, then the probability  $p$  of the happening of event  $E$  is given by:

$$p = \lim_{n \rightarrow \infty} \frac{m}{n}$$

**Axiomatic or Modern Approach:** It was introduced by the Russian mathematician A.N. Kolmogorov (1933) on the basis of set theory. According to this approach, the probability is defined as a set function and is based on certain axioms or postulates given below:

- i) To every outcome of the sample space, a real number  $p$  (called the probability) can be attached and is such that  $0 \leq p \leq 1$ .
- ii) The probability of the entire sample space (sure event) is 1, i.e.  $P(S) = 1$ .
- iii) If  $A$  and  $B$  are mutually exclusive events, then  $P(A \cup B) = P(A) + P(B)$

## 2.2 Laws of Probability:

**Addition Law of Probability:** The probability that atleast one of the two events  $A$  and  $B$  occurs is equal to the probability that event  $A$  occurs, plus the probability that  $B$  occurs minus the probability that both events occur. Symbolically, it can be written as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

However, if the events are mutually exclusive, then the probability of occurrence of event  $A$  or  $B$  is sum of the individual probabilities of  $A$  and  $B$ .

$$\text{Symbolically, } P(A \cup B) = P(A) + P(B)$$

If A, B, C are three events then addition law may be stated as:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Similarly if  $A_1, A_2, \dots, A_k$  are mutually exclusive events then probability of occurrence of either of  $k$  events is given by

$$P(A_1 \cup A_2 \cup \dots \cup A_k) = P(A_1) + P(A_2) + \dots + P(A_k)$$

**Conditional Probability:** It is the probability associated with the events defined on the subset of the sample space. The conditional probability of A given B, is equal to the probability of event A when it is known that another event B has already occurred. Symbolically, we may write it as:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} ; P(B) \neq 0$$

Thus events A and B are independent if and only if  $P(A/B) = P(A)$  and  $P(B/A) = P(B)$  which implies that  $P(A \cap B) = P(A) P(B)$ .

**Multiplication Law of Probability:** If the two events A and B are independent, then the probability that both will occur together is equal to the product of their individual probabilities. Symbolically,

$$P(A \text{ and } B) = P(A \cap B) = P(A) P(B)$$

Similarly if  $A_1, A_2, \dots, A_k$  are independent event then the probability of simultaneous occurrence of  $k$  events is equal to the product of probabilities of their individual occurrence.

$$P(A_1 \cap A_2 \cap \dots \cap A_k) = P(A_1) P(A_2) \dots P(A_k)$$

However, if the events are not independent then the probability of simultaneous occurrence of events A and B is given by:

$$P(A \cap B) = P(A) P(B/A) = P(B) P(A/B)$$

Where  $P(B/A)$  represents the conditional probability of occurrence of B when the event A has already happened and  $P(A/B)$  is similarly defined.

**Joint and Marginal Probability:**

Joint probability is the probability of simultaneous occurrence of two or more events. For example with two events A and B, there are four joint probabilities possible as explained below:

Joint event	Sample Points belonging to the Joint event	Probability
$A \cap B$	$n(A \cap B)$	$n(A \cap B)/n(S)$
$A \cap \bar{B}$	$n(A \cap \bar{B})$	$n(A \cap \bar{B})/n(S)$
$\bar{A} \cap B$	$n(\bar{A} \cap B)$	$n(\bar{A} \cap B)/n(S)$
$\bar{A} \cap \bar{B}$	$n(\bar{A} \cap \bar{B})$	$n(\bar{A} \cap \bar{B})/n(S)$

For example, consider the employment status of skilled and unskilled workers in a village, where the event A implies that the worker is employed and B that he is skilled.

Employment Status	Skilled B	Unskilled ( $\bar{B}$ )	Total
Employed (A)	$n(A \cap B) = 160$	$n(A \cap \bar{B}) = 240$	$n(A) = 400$
Unemployed ( $\bar{A}$ )	$n(\bar{A} \cap B) = 40$	$n(\bar{A} \cap \bar{B}) = 60$	$n(\bar{A}) = 100$
Total	$n(B) = 200$	$n(\bar{B}) = 300$	$n(S) = 500$

The joint probability table is as follows:

Status	B	$\bar{B}$	Marginal Probability
A	$P(A \cap B) = \frac{160}{500} = 0.32$	$P(A \cap \bar{B}) = \frac{240}{500} = 0.48$	$P(A) = 0.80$
$\bar{A}$	$P(\bar{A} \cap B) = \frac{40}{500} = 0.08$	$P(\bar{A} \cap \bar{B}) = \frac{60}{500} = 0.12$	$P(\bar{A}) = 0.20$
Marginal Probability	$P(B) = 0.40$	$P(\bar{B}) = 0.60$	1.00

The row and column probabilities are called marginal probabilities, simply because these are found in the margins of the table. It may also be noted that a marginal probability is the sum of a set of joint probabilities. For example,

$$\begin{aligned}
 P(A) &= \text{Marginal Probability of employed worker} \\
 &= P(A \cap B) + P(A \cap \bar{B}) = 0.32 + 0.48 = 0.80 \\
 P(B) &= \text{Marginal Probability of skilled worker} \\
 &= P(A \cap B) + P(\bar{A} \cap B) = 0.32 + 0.08 = 0.40
 \end{aligned}$$

Also in the above table, we find that

$$\begin{aligned}
 P(A \cap B) &= P(A) \times P(B); P(\bar{A} \cap B) = P(\bar{A}) \times P(B) \\
 P(A \cap \bar{B}) &= P(A) \times P(\bar{B}); P(\bar{A} \cap \bar{B}) = P(\bar{A}) \times P(\bar{B})
 \end{aligned}$$

Thus the joint probabilities are the product of the relevant marginal probabilities which shows that events A and B are independent. In other words, the employment prospects are equal for skilled and unskilled workers.

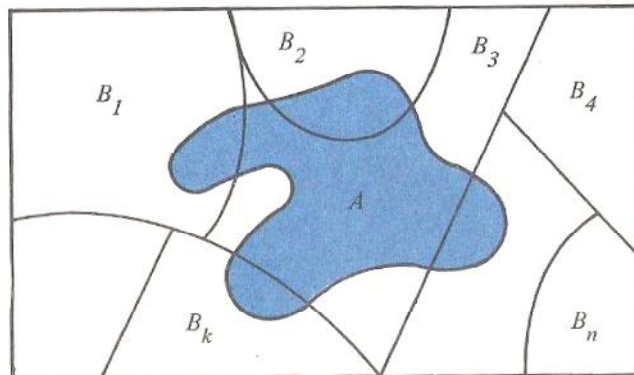
**2.3 Bayes’ Theorem:**

It was introduced by a British Mathematician Thomas Bayes (1702-61). It is useful in the revision of the old (given) probabilities in the light of additional information supplied by the experiment or past records. It may be of extreme help to policy makers in arriving at better decisions in the face of uncertainty. It is given below:

If  $B_1, B_2, \dots, B_n$  be  $n$  mutually exclusive events whose union is the sample space  $\rightarrow S$  and if  $P(B_i) \neq 0, (i = 1, 2, \dots, n)$  then for any arbitrary event A of the sample space  $\rightarrow S$

$$P(B_i/A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(B_i) P(A/B_i)}{\sum_{i=1}^n P(B_i) P(A/B_i)}$$

The probabilities  $P(B_1), P(B_2), \dots, P(B_n)$  are termed as ‘prior probabilities’ because they exist before we gain any information from the experiment.  $P(A/B_i) (i=1, 2, \dots, n)$  are called ‘likelihoods’ because they indicate how likely the event ‘A’ occur given each and every prior probabilities.  $P(B_i/A)$  are called posterior probabilities or revised probabilities.



**Graphical Representation of Bayes’ Theorem**

**Example-1:** A box contains a few red and a few blue balls. One ball is drawn randomly, find the probability of getting a red ball if:

- i) We know that there are 30 red and 40 blue balls in the box.
- ii) We do not know the composition of the box.

**Solution:** Let A be the event of getting a red ball.

When composition of the box is known, we use classical approach to find the required probability and when true composition of the box is unknown, we use the Empirical or relative frequency approach.

i) Composition of the box is known (a priori information)

$$P(A) = \frac{\text{Number of favourable cases to A}}{\text{Exhaustive No. of cases}}$$
$$= \frac{\text{Number of red balls}}{\text{Total number of balls}} = \frac{30}{70} = \frac{3}{7}$$

ii) The experiment of drawing one ball is replicated a large number of times. The relative frequency of the red ball is found, then relative frequency of red ball will be approximately equal to P(A).

If we make n draws with replacement then,

$$P_n(A) = \frac{\text{Number of red balls appeared in n draws}}{n} \approx P(A)$$

**Example-2:** What is the chance that a leap year selected at random will contain 53 Sundays?

**Solution:** A leap year consists of 52 complete weeks with a balance of two days. These two days can be any one of the seven possible days:

- (i) M & T    (ii) T & W    (iii) W & Th    (iv) Th & F  
(v) F & Sat    (vi) Sat & S    (vii) S & M

Out of the seven combinations, only last two are favourable.

$$\therefore \text{Required chance} = 2/7$$

**Example-3:** A basket of fruit contains 3 mangoes, 2 bananas and 7 pineapples. If a fruit is chosen at random, what is the probability that it is either a mango or a banana.

**Solution:** Let  $A_1$ ,  $A_2$  and  $A_3$  respectively denote the events that a chosen fruit is a mango, a banana or a pineapple.

Given

$$\text{Number of mangoes } (m_1) = 3$$

$$\text{Number of bananas } (m_2) = 2$$

Number of pineapples ( $m_3$ ) = 7

$\therefore$  Total number (N) =  $m_1 + m_2 + m_3 = 3 + 2 + 7 = 12$

$\therefore P(A_1) = \frac{m_1}{N} = \frac{3}{12}$ ,  $P(A_2) = \frac{m_2}{N} = \frac{2}{12}$  and  $P(A_3) = \frac{m_3}{N} = \frac{7}{12}$

Since the events  $A_1$ ,  $A_2$  and  $A_3$  are mutually exclusive

P(the chosen fruit is either a mango or a banana) =  $P(A_1 \text{ or } A_2)$

$$= P(A_1) + P(A_2) = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$$

**Example-4:** In a single throw of two dice, determine the prob. of getting

- (a) a total of 2, (b) a total of 12, (c) a total of 7 or 9

**Solution:** Two dice can be thrown in  $6 \times 6 = 36$  ways

- a) A total of 2 can be obtained as  $(1, 1) = 1$  way

$$\therefore P(\text{a total of 2}) = 1/36$$

- b) A total of 12 can be obtained as  $(6, 6) = 1$  way

$$\therefore P(\text{a total of 12}) = 1/36$$

- c) A total of 7 can be obtained as  $(6, 1), (1, 6), (5, 2), (2, 5), (4, 3), (3, 4)$ .

A total of 9 can be obtained as  $(6, 3), (3, 6), (5, 4), (4, 5)$

$\therefore$  a total of 7 or 9 can be obtained in 10 ways.

$$\therefore \text{required Probability} = 10/36 = 5/18.$$

**Example-5:** A card is drawn from a well-shuffled pack of 52 cards. Find the probability of drawing

- (a) a black king  
(b) a jack, queen, king or ace  
(c) a card which is neither a heart nor a king  
(d) a spade or a club.

**Solution:** A card can be drawn from a pack of 52 cards in  ${}^{52}C_1$  ways.

$$(a) P(\text{a black king}) = \frac{{}^2C_1}{{}^{52}C_1} = \frac{2}{52} = \frac{1}{26}$$



$$(b) \quad P(\text{a jack, queen, king or ace}) = \frac{{}^{16}C_1}{{}^{52}C_1} = \frac{16}{52} = \frac{4}{13}$$

$$(c) \quad P(\text{a card which is neither a heart nor a king}) = \frac{{}^{36}C_1}{{}^{52}C_1} = \frac{36}{52} = \frac{9}{13}$$

$$(d) \quad P(\text{a spade or a club}) = \frac{{}^{26}C_1}{{}^{52}C_1} = \frac{26}{52} = \frac{1}{2}$$

**Example-6:** The probability that a boy will get admission in B.Sc. (Ag.) is  $2/3$  and the probability that he will not get admission in M.V.Sc. is  $5/9$ . If the probability of getting admission in at least one course is  $4/5$ , what is the probability that he will get admission in both courses?

Let event A: Boy will get admission in B.Sc. (Ag.)

and B : Boy will get admission in M.V.Sc.

The  $P(A) = 2/3$ ,  $P(B) = 1 - (5/9) = 4/9$ ,  $P(A \cup B) = 4/5$

Thus by application of additive law, we get:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 2/3 + 4/9 - 4/5 = 14/45$$

**Example-7:** A study showed that 65% of managers had a MBA degree and 50% had a B Tech degree. Furthermore, 20% of the managers had a MBA degree but no B Tech degree. What is the probability that a manager selected at random has a MBA degree, given that he has a B Tech degree?

**Solution:**

Let event A : Manager has a MBA degree

B : Manager has a B Tech degree

Then  $P(A) = 0.65$ ,  $P(B) = 0.5$  and  $P(A \cap \bar{B}) = 0.20$ .

We know that  $P(A) = P(A \cap B) + P(A \cap \bar{B}) \Rightarrow P(A \cap B) = P(A) - P(A \cap \bar{B})$

hence  $P(A \cap B) = 0.65 - 0.20 = 0.45$

$$\text{Thus } P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.45}{0.5} = 0.9$$

**Example-8:** In a locality, out of 5,000 people residing, 1200 are above 30 years of age and 3000 are females. Out of 1200 (who are above 30), 200 are females. Suppose, after a

person is chosen you are told that the person is female. What is the probability that the she is above 30 years of age?

**Solution:** Let event A denote the person of age above 30 years and event B that the person is a female. Then,

$$\text{Thus } P(B) = \frac{3000}{5000} = 0.6, \quad P(A \cap B) = \frac{200}{5000} = 0.04$$

$$\text{Hence } P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.04}{0.60} = 0.067$$

**Example-9:** Given two independent events, A, B such that  $P(A) = 0.3$  and  $P(B) = 0.6$ . Determine

- (a) P (A and B),            (b) (A and not B),            (c) P (not A and B),  
 (d) (neither A nor B)    (e) P (A or B)

**Solution:** (a)  $P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B) = 0.3 \times 0.6 = 0.18$

(b)  $P(A \text{ and not } B) = P(A \cap \bar{B}) = P(A) \cdot P(\bar{B}) = 0.3 \times (1 - 0.6)$   
 $= 0.3 \times 0.4 = 0.12$

(c)  $P(\text{not } A \text{ and } B) = P(\bar{A} \cap B) = P(\bar{A}) \cdot P(B) = (1 - 0.3) \times 0.6$   
 $= 0.7 \times 0.6 = 0.42$

(d)  $P(\text{neither } A \text{ nor } B) = P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}) = (1 - 0.3) (1 - 0.6)$   
 $= 0.7 \times 0.4 = 0.28$

(e)  $P(A \text{ or } B) = 1 - P(\text{neither } A \text{ nor } B)$   
 $= 1 - 0.28 = 0.72$

**Example-10:** Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the balls drawn not being replaced.

**Solution:** Let  $A$  be the event that ball drawn is white in first draw.

and  $B$  be the event that ball drawn is white in 2<sup>nd</sup> draw.

$$P(A \cap B) = P(A) \cdot P(B/A)$$

Here  $P(A) = 7/12$  and  $P(B/A) = 6/11$

$$\therefore P(A \cap B) = \frac{7}{12} \times \frac{6}{11} = \frac{7}{22}$$

**Example-11:** The probability of  $n$  independent events are  $p_1, p_2, \dots, p_n$ . Find an expression for the probability that at least one of the events will happen.

**Solution:** Prob. of non-happening of 1<sup>st</sup> event =  $1 - p_1$   
 Prob. of non-happening of 2<sup>nd</sup> event =  $1 - p_2$   
 $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$   
 Prob. of non-happening of  $n$ th events =  $1 - p_n$ .

Since the events are independent

$$\therefore \text{Prob. of non-happening of all the events} \\ = (1-p_1)(1-p_2) \dots (1-p_n)$$

$$\therefore \text{Prob. of happening of at least one of the events.} \\ = 1 - (1-p_1)(1-p_2) \dots (1-p_n)$$

**Example-12:** A problem in statistics is given independently to three students A, B and C whose chances of solving it are  $\frac{1}{2}, \frac{1}{3}$  and  $\frac{1}{4}$  respectively. What is the probability that problem will be solved?

**Solution:** Probability that A solves the problem i.e.  $P(A) = \frac{1}{2}$   
 Probability that B solves the problem i.e.  $P(B) = \frac{1}{3}$   
 Probability that C solves the problem i.e.  $P(C) = \frac{1}{4}$

By addition law of probability:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Since the events are independent

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A) \times P(B) - P(A) \times P(C) - P(B) \times P(C) + P(A) \times P(B) \times P(C) \\ = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} \times \frac{1}{3} - \frac{1}{2} \times \frac{1}{4} - \frac{1}{3} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \\ = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{12} + \frac{1}{24} = \frac{3}{4}$$

**Alternative Method:**

Sometimes, it is more convenient to find the probability of the complement of an event rather than of event itself and this approach will be used here in this example.

Consequently  $P(\bar{A}) = \frac{1}{2}$ ,  $P(\bar{B}) = \frac{2}{3}$  and  $P(\bar{C}) = \frac{3}{4}$

Now probability that problem is solved = 1 - Probability that problem is not solved by any one.

$$\begin{aligned}
 &= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C}) \text{ \{Since events are independent\}} \\
 &= 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C}) \\
 &= 1 - \left[ \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \right] = \frac{3}{4} \text{ Ans.}
 \end{aligned}$$

**Example-13:** A speaks truth in 60% cases and B in 70% cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

**Solution:** It is clear that they will contradict each other only if one of them speaks the truth and other tells a lie.

The probability That A speaks the truth and B tells a lie =  $\frac{60}{100} \times \frac{30}{100} = \frac{9}{50}$

The probability that B speaks the truth and A tells a lie

$$= \frac{70}{100} \times \frac{40}{100} = \frac{14}{50}$$

$\therefore$  The probability that they contradict each other =  $\frac{9}{50} + \frac{14}{50} = \frac{23}{50}$

$\therefore$  The % of cases in which they contradict each other

$$= \frac{23}{50} \times 100 = 46\%$$

**Example-14:** The probability that a man will be alive in 25 years is  $\frac{3}{5}$ , and the probability that his wife will be alive in 25 years is  $\frac{2}{3}$ . Find the probability that

- |                                |                               |
|--------------------------------|-------------------------------|
| a) both will be alive          | b) only the man will be alive |
| c) only the wife will be alive | d) at least one will be alive |
| e) exactly one will be alive   |                               |

**Solution:** Define the events H and W as Husband will be alive in 25 years and Wife will be alive in 25 years respectively.

Given  $P(H) = \frac{3}{5}$  and  $P(W) = \frac{2}{3}$

a)  $P(\text{both will be alive}) = P(H \cap W) = P(H) P(W)$  (since H and W are independent)

$$= \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}$$

b)  $P(\text{only the man will be alive}) = P(H) P(\bar{W}) = P(H) [1 - P(W)] = \frac{3}{5} [1 - \frac{2}{3}] = \frac{1}{5}$

c)  $P(\text{only the wife will be alive}) = P(\bar{H}) P(W) = [1 - P(H)] P(W)$

$$= [1 - \frac{3}{5}] \times \frac{2}{3} = \frac{4}{15}$$

d)  $P(\text{at least one will be alive}) = P(H \cup W) = P(H) + P(W) - P(H \cap W)$

$$= \frac{3}{5} + \frac{2}{3} - \frac{2}{5} = \frac{9 + 10 - 6}{15} = \frac{13}{15}$$

e) Probability that exactly one will be alive i.e. either of the two (not both) =

$$P(H \cap \bar{W}) + P(\bar{H} \cap W) = P(H) \cdot P(\bar{W}) + P(\bar{H}) \cdot P(W)$$

$$= \frac{3}{5} \times \left(1 - \frac{2}{3}\right) + \left(1 - \frac{3}{5}\right) \times \frac{2}{3} = \frac{3}{5} \times \frac{1}{3} + \frac{2}{5} \times \frac{2}{3} = \frac{7}{15}$$

**Example-15:** In a bolt factory machines A, B, and C manufacture respectively 25 per cent, 35 per cent and 40 per cent of the total output. Of the total of their output 5, 4, and 2 per cent are defective bolts, A bolt is drawn at random and is found to be defective. What is the probability that it was manufactured by machines A, B, or C?

**Solution:** Let,  $A_i$  ( $i = 1, 2, 3$ ) be the event of drawing a bolt produced by machine A, B and C, respectively. It is given that

$$P(A_1) = 0.25; P(A_2) = 0.35, \text{ and } P(A_3) = 0.40$$

Let  $B$  = the event of drawing a defective bolt

Thus  $P(B/A_1) = 0.05, P(B/A_2) = 0.04$  and  $P(B/A_3) = 0.02$

By BayesøTheorem, we get:

$$P(A_1/B) = \frac{P(A_1)P(B/A_1)}{\sum_{i=1}^3 P(A_i)P(B/A_i)}$$

$$= \frac{(0.25)(0.05)}{(0.25)(0.05) + (0.35)(0.04) + (0.40)(0.02)} = 0.362$$

Similarly we obtain  $P(A_2/B) = 0.406$  and  $P(A_3/B) = 0.232$

**Second Method:**

**Table of posterior probabilities:**

Event	Prior Probability $P(A_i)$	Conditional Probability $P(B/A_i)$	Joint Probability $(2) \times (3)$	Posterior Probability $P(A_i/B)$
(1)	(2)	(3)	(4)	(5)
$A_1$	0.25	0.05	0.0125	$0.0125 \div 0.0345 = 0.362$
$A_2$	0.35	0.04	0.0140	$0.014 \div 0.0345 = 0.406$
$A_3$	0.40	0.02	0.0080	$0.008 \div 0.0345 = 0.232$
<b>Total</b>	<b>1.00</b>		<b>0.0345</b>	<b>1.000</b>

The above table shows the probability that the given item was defective and produced by machine A is 0.362, by machine B is 0.406, and machine C is 0.232.

**Example-16:** Three urns are given containing white (W), black (B) and red (R) balls as Urn-I: 2W, 3B and 4R; Urn-II: 3W, 1B and 2R balls; Urn-III: 4W, 2B and 5R balls. An urn is selected at random and two balls are drawn from the selected urn and they happen to be one white and one red ball. What is the probability that the balls are drawn from the Urn-II?

**Solution:** Let  $A_1, A_2$  and  $A_3$  be the events that Urn-I, II, III are selected, respectively. Let B be the event that out of two selected balls, one is white and other is red. Thus probabilities of selecting a urn are:

$$P(A_1) = 1/3; P(A_2) = 1/3, \text{ and } P(A_3) = 1/3$$

Probability of selecting one white and one red ball from urn=I is:

$$P(B/A_1) = \frac{{}^2C_1 \times {}^4C_1}{{}^9C_2} = \frac{2 \times 4 \times 2}{9 \times 8} = \frac{2}{9}$$

$$\text{Similarly, } P(B/A_2) = \frac{{}^3C_1 \times {}^2C_1}{{}^6C_2} = \frac{3 \times 2 \times 2}{6 \times 5} = \frac{2}{5}$$

$$\text{And from Urn-III: } P(B/A_3) = \frac{{}^4C_1 \times {}^5C_1}{{}^{11}C_2} = \frac{4 \times 5 \times 2}{11 \times 10} = \frac{4}{11}$$

Now the Bayesøprobability that balls belong to Urn-II is:

$$P(A_2/B) = \frac{P(A_2)P(B/A_2)}{\sum P(A_i)P(B/A_i)} = \frac{1/3 \times 2/5}{\frac{1}{3} \times \frac{2}{9} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{4}{11}} = \frac{99}{244}$$

## 2.4 Random Variable and Probability Distribution:

**Random Variable:** Random variable is a rule which associates uniquely a real number to every elementary event  $e_i \in S$ ,  $i = 1, 2, \dots, n$ . In other words a R.V. is a real valued function defined over the sample space of the experiment i.e. a variable whose values are determined by the outcomes of the experiment.

If the R.V. takes only integer values such as 0, 1, 2, 3, ... then it is called discrete R.V. For example, let X be the number of heads obtained in two independent tosses of a coin. Here  $S = [HH, HT, TH, TT]$  and  $X(HH) = 2$ ,  $X(HT) = 1$ ,  $X(TH) = 1$ ,  $X(TT) = 0$ . Thus X can take values 0, 1, 2, and is a discrete random variable.

Other examples of discrete random variables are: The number of printing mistakes in each page of a book, the number of telephone calls received by the telephone operator.

If the random variable takes on all values with in a certain interval, then it is called a continuous random variable. The amount of rainfall on a rainy day, height and weight of individuals are examples of continuous random variables.

**Probability Distribution:** The rule which assigns probabilities to each of the possible values of a random variable is called a probability distribution.

**Definition:** The distribution of a discrete random variable is called discrete probability distribution.

Let X be a discrete random variable taking values  $p_1, p_2, \dots, p_n$  respectively such that  $\sum p_i = 1$ . Then such a listing of outcomes of X together with their associated probabilities is called discrete probability distribution and can be represented in tabular form as

X	x <sub>1</sub>	x <sub>2</sub>	x <sub>3</sub>	í	..	x <sub>n</sub>	p <sub>i</sub> × 0 and ∑ p <sub>i</sub> = 1
p(x)	p <sub>1</sub>	p <sub>2</sub>	p <sub>3</sub>	í	..	p <sub>n</sub>	

Graphically it can be represented by a set of vertical lines whose heights are proportional to the probabilities p<sub>i</sub> = P(X = x<sub>i</sub>). The function p(x) is called the probability function or probability mass function (p.m.f.) of the discrete R.V.

**Definition:** The distribution of a continuous R.V. is called continuous probability distribution.

It is represented by the curve with equation y = f(x). The probability, that X lies between infinitesimal interval (x, x + dx) is given by f(x)dx where f(x) is called the probability density function (p.d.f.) of the continuous R.V. X with the following properties:

- i) f(x) is non-negative
- ii) Area under the curve y = f(x) and x-axis is unity i.e.  $\int_{-\infty}^{\infty} f(x)dx = 1$
- iii)  $P(a < x < b) = \int_a^b f(x)dx = F(b) - F(a)$

Thus the probability that X lies between any two points a and b is given by the area under the curve y = f(x) from x = a to x = b. The function F(x) = P(X ≤ x) is called the cumulative distribution function (c.d.f.) of X.

**Note:** The area under the curve y = f(x) when a = b is zero and hence P (x = a) = 0. Thus for a continuous R.V.X

$$P (a \leq x \leq b) = P (a < x < b) = P (a \leq x < b) = P (a < x \leq b)$$

**2.5 Expectation, Variance and Standard Deviation (SD):**

If the random variable X assumes the discrete values x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>,.....,x<sub>n</sub> with corresponding probabilities p<sub>1</sub>, p<sub>2</sub>, p<sub>3</sub>,.....,p<sub>n</sub> then we define:

$$\mu = E(X) = p_1x_1 + p_2x_2 + p_3x_3 + \dots + p_nx_n = \sum_{i=1}^n p_i x_i$$

$$\text{Variance} = \sigma^2 = E [x - \mu]^2 = \sum_{i=1}^n p_i (x_i - \mu)^2 = \sum_{i=1}^n p_i x_i^2 - \mu^2$$

$$\text{SD} = \sigma = \sqrt{\text{Var} (X)}$$



In the case of continuous R.V. X with probability density function f(x), we have

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Variance

$$\begin{aligned} \sigma^2 &= E[(x - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx \\ &= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 \\ &= \sqrt{\text{Var}(X)} \end{aligned}$$

**Example-17:** During the course of a day, a machine turns out either 0, 1 or 2 defective pens with probabilities 1/6, 2/3 and 1/6, respectively. Calculate the mean value and the variance of the defective pens produced by the machine in a day.

**Solution:** The probability distribution of number of defective pens (X) is given by:

X	0	1	2
p(x):	1/6	2/3	1/6

$$\therefore E(X) = \sum_{i=0}^2 p_i x_i = (1/6) \times 0 + (2/3) \times 1 + (1/6) \times 2 = 1$$

$$\text{Var}(X) = \sum_{i=0}^2 x_i^2 p_i - \mu^2 = (1/6) \cdot 0^2 + 2/3 \cdot 1^2 + 1/6 \cdot 2^2 - 1^2 = 1/3$$

**Example-18:** Given the following probability distribution

x:	1	2	3	4	5	6	7
p(x):	2λ	2λ	λ	3λ	λ <sup>2</sup>	2λ <sup>2</sup>	7λ <sup>2</sup> + λ

(i) Find λ (ii) Evaluate P(X > 5) and P(X < 4)

**Solution:**

i) Since sum of all probabilities is unity, therefore,

$$2\lambda + 2\lambda + \lambda + 3\lambda + \lambda^2 + 2\lambda^2 + 7\lambda^2 + \lambda = 1$$

or  $10\lambda^2 + 9\lambda - 1 = 0$

or  $(\lambda + 1)(10\lambda - 1) = 0$

$\therefore \lambda = 1/10$  ( $\lambda \neq -1$ , since  $0 \leq p(x) \leq 1$ )

$$\begin{aligned} \text{ii) } P(X \leq 5) &= p(5) + p(6) + p(7) \\ &= \lambda^2 + 2\lambda^2 + (7\lambda^2 + \lambda) = 10\lambda^2 + \lambda \\ &= 10 (1/10)^2 + 1/10 = 1/5 \end{aligned}$$

$$\begin{aligned} \text{and } P(X < 4) &= p(1) + p(2) + p(3) \\ &= 2\lambda + 2\lambda + \lambda = 5\lambda \\ &= 5 \times (1/10) = 1/2 \end{aligned}$$

**Example-19:** Find the mean number of heads in three tosses of a coin.

**Solution:** Let  $X$  denote a random variable which is the no. of heads obtained in three tosses of a coin.

Clearly  $X$  takes the values 0, 1, 2, 3. Let  $p$  and  $q$  be the probabilities of turning up head and tail respectively:

$$p = 1/2, q = 1 - 1/2 = 1/2$$

$$\therefore P(X=0) = 1/2 \times 1/2 \times 1/2 = 1/8$$

$$P(X=1) = 3 \times (1/2 \times 1/2 \times 1/2) = 3/8$$

$$P(X=2) = 3 \times (1/2 \times 1/2 \times 1/2) = 3/8$$

$$P(X=3) = 1/2 \times 1/2 \times 1/2 = 1/8$$

$x_i$	$p_i$	$p_i x_i$
0	1/8	0
1	3/8	3/8
2	3/8	6/8
3	1/8	3/8

$$\text{Mean} = \sum p_i x_i = 12/8 = 3/2 \quad \therefore \mu = 3/2$$

**Properties of Expectation:**

i) For a constant  $a$ ,  $E(a) = a$  and  $E(aX) = aE(X)$

ii) **Addition Theorem of Expectation:** If  $X_1, X_2, \dots, X_n$  are  $n$  RVs then

$$E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$$

i.e. the expectation of sum of several RVs is equal to the sum of their individual expectations.

iii) **Multiplication Theorem of Expectation:** If  $X_1, X_2, \dots, X_n$  are  $n$  independent RVs then

$$E(X_1 X_2 \dots X_n) = E(X_1) \cdot E(X_2) \cdot \dots \cdot E(X_n)$$

i.e. the expectation of product of several independent R.Vs is equal to the product of their individual expectations.

(iv) If  $X_1, X_2, \dots, X_n$  are  $n$  variables and  $a_1, a_2, \dots, a_n$  are  $n$  constants. Then the expectation of linear combination of  $X_i$ s given by

$$E(a_1 X_1 + a_2 X_2 + \dots + a_n X_n) = a_1 E(X_1) + a_2 E(X_2) + \dots + a_n E(X_n)$$

**Properties of Variance:**

i) If  $a$  is a constant then  $V(a) = 0$  and  $V(aX) = a^2 V(X)$

ii) If  $X_1, X_2$  are two RVs and  $a_1, a_2$  are constants then

$$V(a_1 X_1 + a_2 X_2) = a_1^2 V(X_1) + a_2^2 V(X_2) + 2a_1 a_2 \text{Cov}(X_1, X_2)$$

iii) If  $X_1, X_2$  are independent then since  $\text{Cov}(X_1, X_2) = 0$ , so that

$$V(a_1 X_1 + a_2 X_2) = a_1^2 V(X_1) + a_2^2 V(X_2)$$

When  $a_1 = a_2 = 1$ , we get  $V(X_1 + X_2) = V(X_1) + V(X_2)$  which can further be generalized i.e. if  $X_1, X_2, \dots, X_n$  are  $n$  independent random variables, then

$$V(X_1 + X_2 + \dots + X_n) = V(X_1) + V(X_2) + \dots + V(X_n)$$

Hence variance of the sum of several independent RVs is equal to the sum of their individual variances.

iv) If  $X_1, X_2, \dots, X_n$  are  $n$  independent RVs then variance of linear combination of several independent variable is

$$V(a_1 X_1 + a_2 X_2 + \dots + a_k X_n) = a_1^2 V(X_1) + \dots + a_n^2 V(X_n)$$

**Co-variance in Terms of Expectation:** Simultaneous variation in the two variables is called co-variance. It tells us how the two variables move together. If  $X_1$  and  $X_2$  are two RVs with  $E(X_1) = \mu_1$  and  $E(X_2) = \mu_2$  then covariance of  $X_1$  and  $X_2$  is given by

$$\begin{aligned} \text{Cov}(X_1, X_2) &= E[X_1 - E(X_1)][X_2 - E(X_2)] \\ &= E(X_1 - \mu_1)(X_2 - \mu_2) \end{aligned}$$

If  $p_i = 1/n; i = 1, 2, \dots, n$  then

$$\text{Cov}(X_1, X_2) = \frac{1}{n} \sum X_1 X_2 - \mu_1 \mu_2$$

Covariance of  $X_1$  and  $X_2$  denotes how the two variables vary together. Further if  $X_1$  and  $X_2$  are independent then

$$\begin{aligned} \text{Cov}(X_1, X_2) &= E[X_1 \delta E(X_1)] \cdot [X_2 \delta E(X_2)] \\ &= E[X_1 \delta E(X_1)] \cdot E[X_2 \delta E(X_2)] \\ &= [E(X_1) \delta E(X_1)] \cdot [E(X_2) \delta E(X_2)] = 0 \end{aligned}$$

Hence if the two RVs are independent then their covariance is zero but the converse is not true.

**Example-20:** If  $X$ ,  $Y$  and  $Z$  are three independent variables with  $E(X) = 3$ ,  $E(Y) = -4$  and  $E(Z) = 6$ ,  $\sigma_x^2 = 4$ ,  $\sigma_y^2 = 9$  and  $\sigma_z^2 = 1$ . Find the mean and variance of  $L = 2X + 3Y \delta Z + 8$

**Solution:**  $L = 2X + 3Y \delta Z + 8$

Mean of  $L = E(L) = 2E(X) + 3E(Y) \delta E(Z) + E(8)$

$$\begin{aligned} &= 2 \times 3 + 3 \times (-4) - 6 + 8 \\ &= 6 - 12 \delta 6 + 8 = -4 \end{aligned}$$

$$\begin{aligned} V(L) &= 2^2 V(X) + 3^2 V(Y) + (-1)^2 V(Z) + V(8) \\ &= 4 \times 4 + 9 \times 9 + 1 \times 1 + 0 \\ &= 16 + 81 + 1 + 0 = 98 \end{aligned}$$

**Example-21:** If  $X_1$ ,  $X_2$  and  $X_3$  are dependent random variables and having means 4, 8, 6 and variances 5, 8, 6 and  $\text{Cov}(X_1, X_2) = 2$ ,  $\text{Cov}(X_1, X_3) = 3$  and  $\text{COV}(X_2, X_3) = 6$ . Find the mean and variance of  $Y = 2X_1 \delta 3X_2 + 3X_3$

**Solution:** Mean of  $Y = E(Y) = E(2X_1 - 3X_2 + 3X_3)$

$$\begin{aligned} E(Y) &= 2E(X_1) + (-3)E(X_2) + 3E(X_3) \\ &= 2 \times 4 \delta 3 \times 8 + 3 \times 6 \\ &= 8 \delta 24 + 18 = 2 \end{aligned}$$

If  $L = a_1X_1 + a_2X_2 + a_3X_3$ , then

$$\begin{aligned} V(L) &= a_1^2 V(X_1) + a_2^2 V(X_2) + a_3^2 V(X_3) + 2 a_1 a_2 \text{Cov}(X_1, X_2) + \\ &\quad 2 a_2 a_3 \text{Cov}(X_2, X_3) + 2 a_1 a_3 \text{Cov}(X_1, X_3) \end{aligned}$$

$$\begin{aligned} V(Y) &= 2^2 \times 5 + (-3)^2 \times 8 + (3)^2 \times 6 + 2 \times 2 \times (-3) \times 2 + 2(-3) \times 3 \times 6 + 2 \times 2 \times 3 \times 3 \\ &= 20 + 72 + 54 \delta 24 \delta 108 + 36 = 50 \end{aligned}$$

**EXERCISES**

1. An Urn contains 8 red, 7 white and 3 black balls. A ball is drawn at random. What is the probability that the ball drawn will be (i) red (ii) white or black (iii) red or black?
2. One candidate applies for a job in two firms A and B. the probability of his being selected in firm A is 0.7 and being rejected in B is 0.5. The probability that at least one of his applications being rejected is 0.6, what is the probability that he will be selected in one of the firms?
3. A market survey conducted in four cities pertained to preference for brand A soap. The responses are shown below:

	<b>Delhi</b>	<b>Kolkata</b>	<b>Chennai</b>	<b>Mumbai</b>
Yes	45	55	60	50
No	35	45	35	45
No option	5	5	5	5

- What is the probability that a consumer selected at random (i) preferred brand A (ii) preferred brand A and was from Chennai (iii) preferred brand A given that he was from Chennai (iv) given that a consumer preferred brand A, what is the probability that that he was from Mumbai.
4. A problem in business statistics is given to five students A, B, C, D and E whose chances of solving these are  $1/2$ ,  $1/3$ ,  $1/4$ ,  $1/5$  and  $1/6$  respectively. What is the probability that the problem will be solved.  
[Hint  $P(\text{problem will be solved}) = 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C}) \cdot P(\bar{D}) \cdot P(\bar{E})$ ] [ Ans.  $5/6$ ]
  5. A husband and wife appear in an interview for two vacancies on the same post. The probability of husband's selection is  $1/7$  and that of wife's selection is  $1/5$ . What is the probability that (a) both will be selected (b) only one will be selected (c) none will be selected [Ans.  $1/35$ ,  $2/7$ ,  $24/35$ ]
  6. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter, car and a truck is  $1/100$ ,  $3/100$  and  $3/20$  respectively. One of the insured persons meets with an accident. What is the probability that he is a (1) scooter driver (2) car driver (3) truck driver? [Ans.  $1/52$ ,  $3/26$ ,  $45/52$ ]
  7. Three urns are given, each containing red and black balls as urn 1; 6 red and 4 black balls urn 2; 2 red and 6 black balls urn 3; 1 red and 8 black balls. An urn is chosen at random and a ball is drawn from the urn and found to be red. Find the probability that the ball is drawn from urn 2 or urn 3 [Ans.  $65/173$ ]